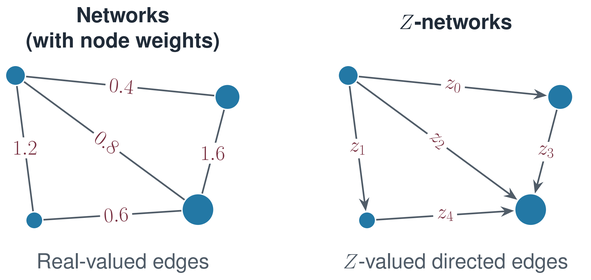
The Z-Gromov-Wasserstein Distance

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From networks to Z-networks [1]

Replace real-valued edge weights by values in a metric space (Z, d_Z).



Real-valued edges Z-valued directed edges

Formal definition [1]

A Z**-network** is a triple

$$\mathsf{X} = (X, \omega_X, \mu_X), \qquad \omega_X : X \times X \longrightarrow Z.$$

X is a Polish space, μ_X a Borel probability measure, and ω_X a measurable, p-integrable function.

The measure describes node mass. The kernel records the directed relation between each pair of nodes.

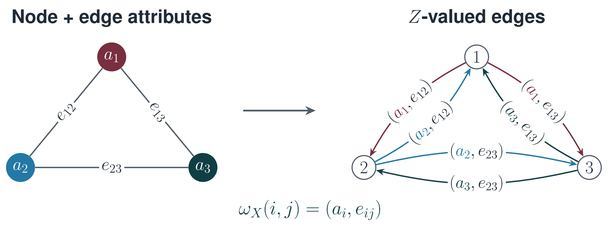
Many objects, one representation [1]

The examples of Z-networks include the following objects.

Metric measure spaces : $Z = \mathbb{R}$

The special case when ω_X is a distance metric.

Attributed graphs: $Z =$ product of node & edge feature spaces



$$\omega_X(i,j) = (a_i, e_{ij})$$

Copy each node attribute onto every outgoing edge.

Shape graphs: $Z =$ the space of curves

Edges represent a curve (e.g., traffic networks, brain arterial network)

Probabilistic networks: $Z =$ the space of probabiliy distributions

Edges carry probability distributions of the edge values.

One theory, *many distances*

A common theoretical framework for comparing structured data, modeled as Z**-networks**

Existing GW distances

Wasserstein	Standard GW	Fused GW
Spectral GW	Ultrametric GW	Fused network GW

unify theory

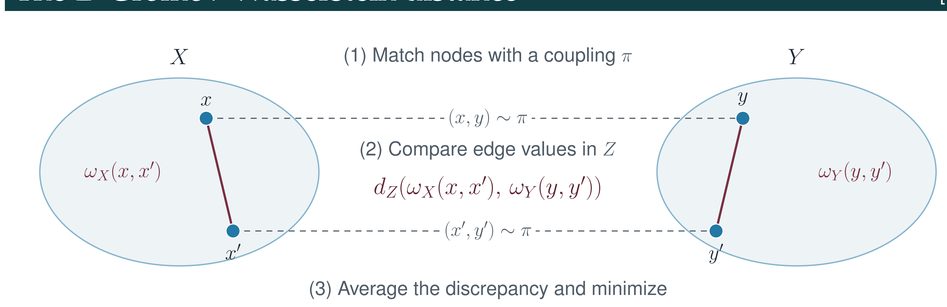
$$Z\text{-GW}$$

define new distances

Existing objects

Shape graphs	Connection graphs	Probabilistic networks

The Z-Gromov-Wasserstein distance [1]

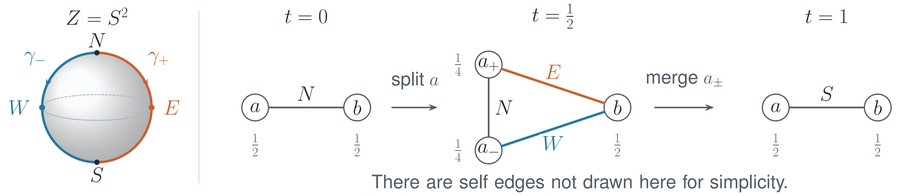


$$\mathrm{GW}_p^Z(\mathsf{X},\mathsf{Y}) = \frac{1}{2} \inf_{\pi \in \Pi(\mu_X,\mu_Y)} \left(\iint d_Z(\omega_X(x,x'), \omega_Y(y,y'))^p \, \mathrm{d}\pi(x,y) \, \mathrm{d}\pi(x',y') \right)^{1/p}.$$

$\Pi(\mu_X, \mu_Y)$ contains couplings of the node measures. Here $1 \le p < \infty$.

Interpolation of Z-networks (geodesics): interpolate edge values [2]

A path of Z-networks X_t, $t \in [0,1]$ is a **geodesic** if $\mathrm{GW}_p^Z(\mathsf{X}_s, \mathsf{X}_t) = |s-t| \, \mathrm{GW}_p^Z(\mathsf{X}_0, \mathsf{X}_1)$. For $1 < p < \infty$ and geodesic Z, every geodesic admits a generalized interpolation: a GW_p^Z-optimal coupling matches nodes, and edge values move along geodesics in Z. Nodes may split to follow different geodesics, as illustrated by γ_+ and γ_-.



There are self edges not drawn here for simplicity.
The features on self edges are N at $a, a_\pm$ and S at b.

A chain of submetries [2]

A mapping $f : X \to Y$ between metric spaces X, Y is a **submetry** when it maps balls to balls: $f(B_X(x,r)) = B_Y(f(x), r)$. Submetries are a metric generalization of Riemannian submersions.

For Polish Z and $1 \le p < \infty$:

Kernels		Z-networks		Distributions
$L^p(I^2, Z)$	$\xrightarrow{Q}$	$\mathcal{M}_p(Z)$	$\xrightarrow{E}$	$\mathcal{P}_p(Z)$
$\frac{1}{2} D_p^Z$		GW_p^Z		$\frac{1}{2} W_p^Z$

$I = [0,1]$ with Lebesgue measure

Q is the quotient map from the L^p from its quotient, i.e., the GW space. $E(X, \omega_X, \mu_X) = \omega_*(\mu_X \otimes \mu_X)$ is the distribution of edge values.

Submetry propagation

The submetry chain propagates completeness, geodesics, and lower curvature bounds from left to right.

Theorem. (Completeness and geodesics) [2]
Completeness of Z is equivalent to completeness of all three spaces. For $1 < p < \infty$, the same holds for geodesicity. When $p = 1$, all three spaces are geodesic for every Polish Z.

Curvature: the special role of $p = 2$ [2]

The **curvature** in the sense of Alexandrov describes how triangles in the space look: a positive curvature makes the triangles "fatter" than in the Euclidean plane, while a negative curvature makes them "thinner". Lower curvature bounds support geometric tools for studying **gradient flows** to optimize functionals on these spaces and compute network **barycenters**.

Theorem. (Curvature bounds) [2]
Let Z be geodesic and Polish with at least two points, (M, μ) a nonatomic Polish probability space, and $1 \le p < \infty$.

Case / bound	Lebesgue $L^p(M,Z)$	Z-GW $\mathcal{M}_p(Z)$	Wasserstein $\mathcal{P}_p(Z)$
$p = 2$ Lower	0 if Z is CBB(0); $-\infty$ otherwise	0 if Z is CBB(0); $-\infty$ otherwise	0 if Z is CBB(0); $-\infty$ otherwise
$p = 2$ Upper	0 if Z is CAT(0); $+\infty$ otherwise	$+\infty$	0 if $Z \cong$ interval; $+\infty$ otherwise
$p \neq 2$ Lower / upper	$-\infty$ / $+\infty$	$-\infty$ / $+\infty$	$-\infty$ / $+\infty$

$-\infty$ / $+\infty$ mean no finite lower / upper bound, respectively. CBB(0) / CAT(0) means curvature ≥ 0 / ≤ 0, respectively.

Papers

[1] Bauer, Mémoli, Needham, Nishino. *The Z-Gromov-Wasserstein Distance.* JMLR, 2025.
[2] Bauer, Mémoli, Needham, Nishino. *Metric Geometry of Lebesgue, Wasserstein, and Gromov-Wasserstein Spaces: Submetries, Curvature, and Geodesics.* arXiv:2608.11680, 2026.

JMLR paper