Identification and Bounding of Joint Expectation over Potential Outcomes

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Abstract.

- ✓ The joint expectation of functions of potential outcomes (Y_1, Y_0) is a fundamental quantity in causal inference, particularly when investigating **the variation or heterogeneity of causal effects**.
- ✓ This paper provides **novel assumptions for their identification**: comonotonicity and countercomonotonicity. These assumptions offer a unified framework for their identification **across discrete, continuous, and mixed outcome variables**.
- ✓ In the absence of these assumptions, we derive **sharp bounds for two broad classes of functions**, yielding new results for bounding moments of individual causal effects, which are central to measuring effect heterogeneity.
- ✓ We present corresponding estimation methods and illustrate them in simulation studies and real-world datasets.

Notation.

$X \in \{0,1\}$: Treatment (binary), Y: Outcome, SCM $\mathcal{M}$: $Y := f_Y(X, U_Y)$, Y_x: Potential Outcome (PO) under treatment x

- ➢ **Individual Causal Difference Effect (ICDE):** $\text{ICDE} = Y_1 - Y_0$
- ➢ **Individual Causal Ratio Effect (ICRE):** $\text{ICRE} = Y_1/Y_0$

Joint Expectations over POs.

➢ **Joint expectations over POs:** $\mathbb{E}[g(Y_1, Y_0)] = \int g(y_1, y_0) dF_{(Y_1,Y_0)}(y_1, y_0)$

What does this class include?

➢ **Distribution of ICDE or ICRE**

$$\mathbb{P}(Y_1 - Y_0 < y) = \int \mathbb{I}(y_1 - y_0 < y) dF_{(Y_1,Y_0)}(y_1, y_0), \mathbb{P}(Y_1/Y_0 < y) = \int \mathbb{I}(y_1/y_0 < y) dF_{(Y_1,Y_0)}(y_1, y_0)$$

➢ **Moments of ICDE or ICRE**

1. **Average of ICDE:** $\mathbb{E}[Y_1 - Y_0]$ (Joint Distribution is not needed.)
2. **Average of ICRE:** $\mathbb{E}[Y_1/Y_0]$
3. **Variance of ICDE or ICRE:** $\text{Var}[Y_1 - Y_0], \text{Var}[Y_1/Y_0]$
4. **Skewness and Kurtosis of ICDE and ICRE**
5. **Central Moments of ICDE and ICRE:** $\mathbb{E}[\{(Y_1 - Y_0) - \mathbb{E}[Y_1 - Y_0]\}^m], \mathbb{E}[\{(Y_1/Y_0) - \mathbb{E}[Y_1/Y_0]\}^m]$

✓ These moments reveal the shape of the distribution of individual causal effects and, with it, the degree of causal effect heterogeneity.

Identification of Joint Expectations over POs.

Several assumptions can be used to identify the joint distribution of potential outcomes.

Table 1: Summary of Identification Assumptions and Their Objectives.

Research	Outcome Type	Assumption	Objective
p. 504 in [Heckman et al., 1997]	Continuous	Rank Preservation	Values of Unobserved POs
Assumption 4.6 in [Wu et al., 2025]	Mixed	Rank Preservation with $\tilde{\rho}$	Values of Unobserved POs
Assumption of Theorem 1 in [Xie et al., 2023]	Continuous	Strict monotonicity over f_Y	Values of Unobserved POs
Definition 14 in [Tian and Pearl, 2000]	Binary	Monotonicity	PoC
Assumption 2 in [Kawakami and Tian, 2025]	Continuous	Monotonicity over f_Y	Moments of ICDE
Assumption 1 and 2 in this paper	Mixed	Comonotonicity and Countercomonotonicity	Joint Expectations over POs

➢ We introduce **comonotonicity and countercomonotonicity assumptions.**

Assumption: 1. Potential outcomes (Y_1, Y_0) are **comonotonic**.
2. Potential outcomes (Y_1, Y_0) are **countercomonotonic**.

1. **Comonotonicity** means that whenever individual A would have a higher outcome than B under $do(X = 0)$, A would have a higher outcome than B under $do(X = 1)$, almost surely.
2. **Countercomonotonicity** is the reverse of comonotonicity

✓ These assumptions offer a unified framework for their identification **across discrete, continuous, and mixed outcome variables**.

We write

$$F^U_{(Y_1,Y_0)}(y_1, y_0) = min\{F_{Y_0}(y_0), F_{Y_1}(y_1)\}, F^L_{(Y_1,Y_0)}(y_1, y_0) = max\{F_{Y_0}(y_0) + F_{Y_1}(y_1) - 1, 0\}$$

$$\mathbb{E}^U[g(Y_1, Y_0)] = \int g(y_1, y_0) dF^U_{(Y_1,Y_0)}(y_1, y_0), \mathbb{E}^L[g(Y_1, Y_0)] = \int g(y_1, y_0) dF^L_{(Y_1,Y_0)}(y_1, y_0)$$

Theorem. Under SCM $\mathcal{M}$, given F_{Y_0} and F_{Y_1}, we have the following two identification statements:
(A). Under Assumption 1 (comonotonicity), $\mathbb{E}[g(Y_1, Y_0)] = \mathbb{E}^U[g(Y_1, Y_0)]$.
(B). Under Assumption 2 (countercomonotonicity), $\mathbb{E}[g(Y_1, Y_0)] = \mathbb{E}^L[g(Y_1, Y_0)]$.

Bounding Joint Expectations over POs.

We study the bounding problem for two important classes.

Theorem. Under SCM $\mathcal{M}$, given F_{Y_0} and F_{Y_1}, we have the following two statements:
(A). The function $g(y_1, y_0)$ is bounded and **supermodular,** i.e., the inequality $g(y'_1, y'_0) - g(y_1, y'_0) \geq g(y'_1, y_0) - g(y_1, y_0)$ holds for every y_0, y_1, y'_0, y'_1 such that $y_1 \leq y'_1$ and $y_0 \leq y'_0$,

$$\mathbb{E}^L[g(Y_1, Y_0)] \leq \mathbb{E}[g(Y_1, Y_0)] \leq \mathbb{E}^U[g(Y_1, Y_0)].$$

(B). The function $g(y_1, y_0)$ is bounded and **submodular,** i.e., the inequality $g(y'_1, y'_0) - g(y_1, y'_0) \leq g(y'_1, y_0) - g(y_1, y_0)$ holds for every y_0, y_1, y'_0, y'_1 such that $y_1 \leq y'_1$ and $y_0 \leq y'_0$,

$$\mathbb{E}^U[g(Y_1, Y_0)] \leq \mathbb{E}[g(Y_1, Y_0)] \leq \mathbb{E}^L[g(Y_1, Y_0)].$$

✓ Combining these bounds yields bounds on causal quantities, which are **sharp** in some cases.

An Application.

Lalonde dataset: The treatment variable is a binary indicator (1 = treated, 0 = control). We use real earnings in 1978 as the outcome variable. Table shows the estimates under comonotonicity (Ours) and rank preservation (RP).

Results: The mean ICDE is positive, but the ICDE and ICRE show substantial heterogeneity: both are positively skewed and highly kurtotic, so there is tail risk. Rank preservation introduces bias with this mixed outcome.

Statistics for $Y_1 - Y_0$ and Y_1/Y_0	Estimates
Mean of ICDE	840.603 [-261.218,1782.534]
Variance of ICDE (Ours)	4.299×10^6 [4.688×10^5 ,1.236×10^7]
Variance of ICDE (RP)	2.929×10^6 [2.993×10^5 ,1.049×10^7]
Skewness of ICDE (Ours)	5.903 [-2.422 ,11.735]
Skewness of ICDE (RP)	4.983 [-7.903,13.555]
Kurtosis of ICDE (Ours)	76.837[5.483,204.682]
Kurtosis of ICDE (RP)	63.362 [4.905,227.056]
Mean of ICRE (Ours)	50.782 [5.957,139.223]
Mean of ICRE (RP)	1.353[1.054,1.680]
Variance of ICRE (Ours)	4.903×10^3 [2.898×10^3 ,2.221×10^5]
Variance of ICRE (RP)	2.420 [0.041,9.0247]
Skewness of ICRE (Ours)	4.497 [2.741,8.634]
Skewness of ICRE (RP)	9.415 [4.685,14.411]
Kurtosis of ICRE (Ours)	29.678 [9.915,89.070]
Kurtosis of ICRE (RP)	100.260 [25.990,202.785]

Conclusion.

- ✓ We study the identification and bounding of the joint expectation $\mathbb{E}[g(Y_1, Y_0)]$ over POs.
- ✓ Although we obtain sharp bounds for certain moments of ICDE and ICRE, our general results are restricted to specific classes of g functions. Extending sharp bounds to more general function classes is future work.