Low Degree Hardness on Broadcasting on Trees

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NEURAL INFORMATION PROCESSING SYSTEMS

Abstract

We study the low-degree hardness of broadcasting on trees. Broadcasting on trees has been extensively studied in statistical physics, in computational biology in relation to phylogenetic reconstruction and in statistics and computer science in the context of block model inference, and as a simple data model for algorithms that may require depth for inference.

The inference of the root can be carried by celebrated Belief Propagation (BP) algorithm which achieves Bayes-optimal performance. Recent works indicated that this algorithm in fact requires high level of complexity.

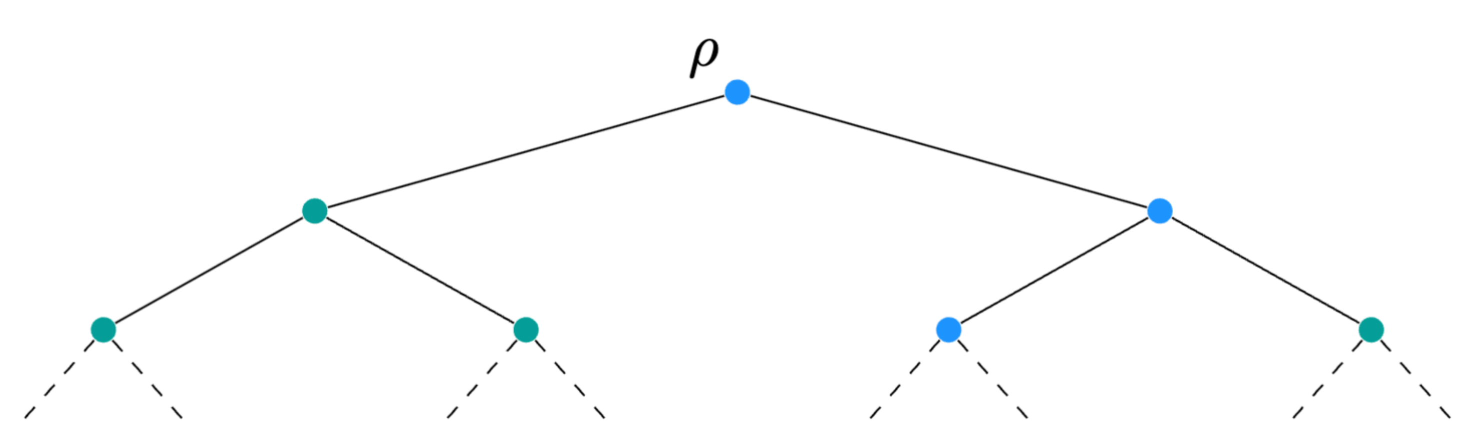
In this work, we prove that this is indeed the case for low degree polynomials. We show that for the broadcast problem using any Markov chain on trees with N leaves, below the Kesten-Stigum bound, any $\log(N)$ degree polynomial has vanishing correlation with the root.

Our result is one of the first low-degree lower bound that is proved in a setting that is not based or easily reduced to a product measure.

Broadcasting on Tree Models

A Simple Example of Broadcasting on Tree (BOT) Models

Consider a rooted tree of degree d and depth ℓ. We describe a board casting process on this tree as follows: Begin by coloring the root vertex ρ with either blue or green. For each child vertex of a colored vertex, the same color is assigned with a probability of 0.7, while the opposite color is assigned otherwise. This process continues iteratively, resulting in a coloring of all the vertices in the tree.



In this example, we initialize the root with blue color, and passing the same color with 70% probability. There are 2^ℓ leaves with are colored by blue or green colors. If we only observe the leave colors, could we infer the root color?

In the general model, we can have q states (colors), and also a $q \times q$ transition matrix M, where the entry M_{ab} is the probability that the color of the child vertex will be b given that the parent vertex has color a. Further, d could also be the average number of children for each vertex, rather than the a fixed number.

Can you infer the root color from the leaves colors?

The Reconstruction problem asks whether one could infer the root color from the leaves colors (bottom layer) more accurately than a random guess as the depth $\ell \to \infty$.

To be precise, we are asking if there exists a (sequence of) function f with input X_L, the leave colors, such that the probability $f(X_L)$ equals to the root will be strictly better than a random guess even as $\ell \to \infty$. We say it is *reconstructable* if this is satisfied.

A fundamental result in this area is the *Kesten-Stigum* bound [KS66]

$$d\lambda^2 = 1$$

where λ is the second largest magnitude of eigenvalues of M, denoted by λ.

It was known to be a threshold for *counting reconstruction*, which means inferring the root value solely by counting the leaves of each color. When $d\lambda^2 > 1$, counting the leaves colors can reconstruct the root value. When $d\lambda^2 < 1$, it is known that, in some simple examples like the one presented on the left, it is impossible to recover the root value better than a random guess for any function (beyond counting) of the leave colors. Nevertheless, there are also models where it is known that, from the information-theoretical perspective, there are more complicated function than counting which can reconstruct the root value, such as Belief Propagation.

Statistical Computational Gaps and Low-degree Framework

The *statistical-computational gap* refers to a regime of a problem is solvable from the statistical perspective. However, it cannot be solved efficiently.

A very exciting line of work, including [HS17, Hop18, KWB19, BKW20, GJW20, MW21, HW20, BH21, Wei20] recently showed that the *low-degree heuristic* can be used to predict computational-statistical gaps for a variety of problems such as recovery in general stochastic block models, sparse PCA, tensor PCA, the planted clique problem, certification in the zero-temperature Sherrington-Kirkpatrick model, the planted sparse vector problem, and for finding solutions in random k-SAT problems.

Here, the (Efron-Stein) degree of a function f with many variables is $\leq k$ if we can express it as a sum of functions $f = \sum f_\alpha$ such that each summand function involves only k variables.

In the BOT model, above the Kesten-Stigum threshold, counting leave colors can infer the root value and only has degree 1. When $\lambda = 0$ (or $d\lambda^2 = 0$), [KM22] show that any function with degree less than $\exp(c\ell)$ has vanishing correlation with the root value, meaning that one cannot infer the root as $\ell \to \infty$ with such polynomial, which indicates there is a statistical-computational gap. They also conjecture that the Kesten-Stigum bound is a threshold for the low degree framework. Here we give a positive answer:

Theorem (informal)

Consider BOT on a rooted tree with average degree bounded by d. If $d\lambda^2 < 1$, any (sequence of) function of the leaves of degree less than $c\ell$ has vanishing correlation with the root value as $\ell \to \infty$, where $c > 0$ is a fixed constant.

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