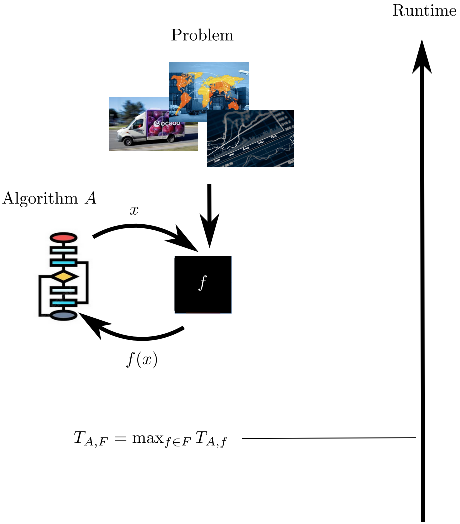
No Free Lunch Theorem and Black-Box Complexity Analysis for Adversarial Optimisation

Per Kristian Lehre and Shishen Lin

School of Computer Science, The University of Birmingham,
Birmingham, United Kingdom

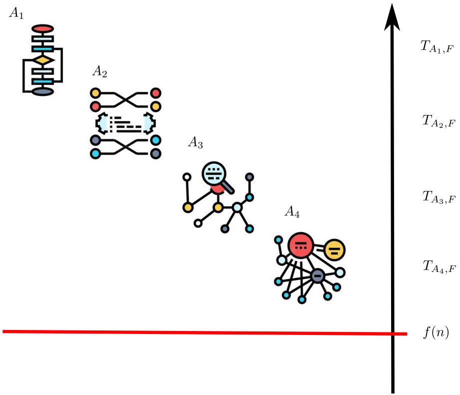
November, 2024

Runtime

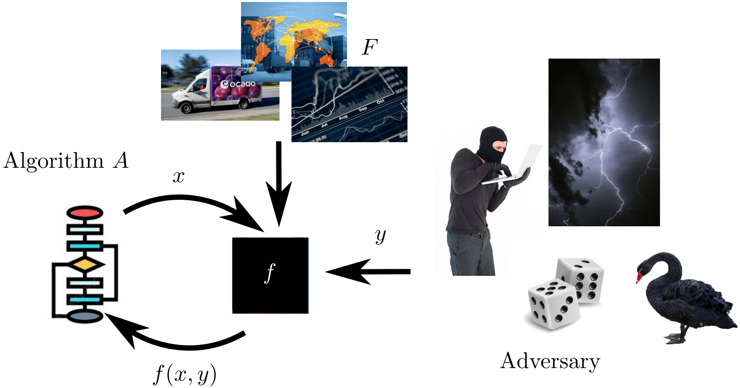


Given a problem class F, $T_{A,F}$ is the worst-case runtime of algorithm A.

Lower Bounds



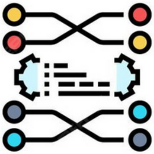
Given a problem size $n \in \mathbb{N}$, for all algorithms A, $T_{A,F} \geq f(n)$.


F
Algorithm A
x
f
y
$f(x,y)$
Adversary

Adversarial No Free Lunch

Algorithm A

Algorithm B



Theorem (Informal)

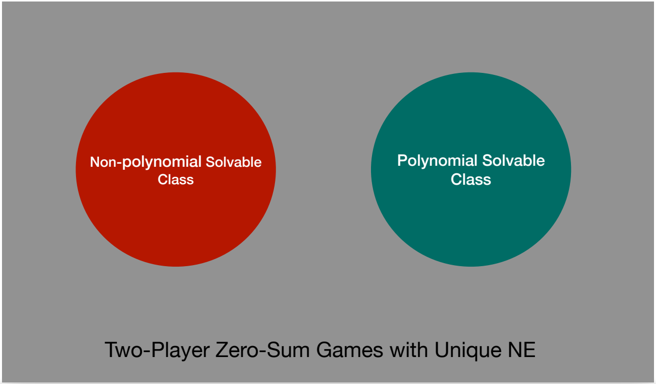
If F is a class of games closed under permutation, then for any algorithms A and B [a]*,*

$$\mathbf{E}_{f \sim \mathrm{Unif}(F)} \left[T_{A,f} \right] = \mathbf{E}_{f \sim \mathrm{Unif}(F)} \left[T_{B,f} \right]$$

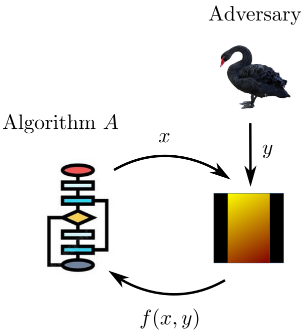
[a] We use the different query model compared with previous work [WM05].

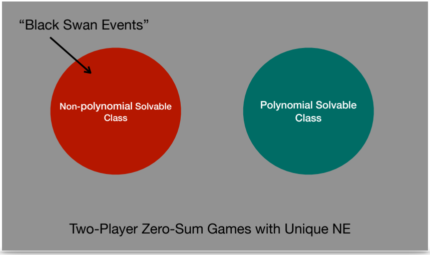
Polynomial-Solvable vs Non-Polynomial Solvable Class

How can we classify the problem class in two-player zero-sum games with unique NE?



Adversarial Black Box Lower Bound: Black Swan





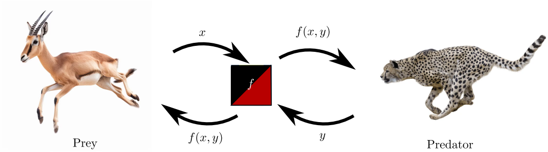
Theorem (Informal)

Consider any game $f : \{0,1\}^n \times \{0,1\}^n \to \mathbb{R}$ with a unique Pure Nash equilibrium where

$$\left|\sum_{i=1}^{n} y_i - \frac{n}{2}\right| \leq \varepsilon n \quad \text{and} \quad \left|\sum_{i=1}^{n} y_i' - \frac{n}{2}\right| \leq \varepsilon n \quad \text{implies} \quad f(x,y) = f(x,y').$$

Let $F = \{f(x, y \oplus z) \mid z \in \{0,1\}^n\}$. Then all algorithms A have runtime $T_{A,F} = e^{\Omega(n)}$.

Co-Evolutionary Arms Races



Theorem

The DIAGONAL *game has black-box complexity* $\Theta(n)$.

Conclusion

Take-home message: You can benefit from competition, but only if the game has a certain structure

- if the game has no structure (closed under permutation),
  then your choice of algorithm does not matter (No Free Lunch).
- all algorithms are inefficient against games with "black swan" structure
- there are games that allow efficient algorithms through co-evolutionary arms races.

End of the Presentation

Thank you for your attention!
Welcome to our poster presentation!
Any questions?

(a) Paper (b) Me (c) Group

Figure: Theory of Evolutionary Computation UoB at Lisbon

References I

Sébastien Bubeck et al.
Convex Optimization: Algorithms and Complexity.
Foundations and Trends® in Machine Learning, 8(3-4):231–357, 2015.

Alistair Benford and Per Kristian Lehre.
Runtime Analysis of Coevolutionary Algorithms on a Class of Symmetric Zero-Sum Games.
In *GECCO'24: Genetic and Evolutionary Computation Conference*. Association for Computing Machinery (ACM), 2024.

Yang Cai, Haipeng Luo, Chen-Yu Wei, and Weiqiang Zheng.
Uncoupled and Convergent Learning in Two-Player Zero-Sum Markov Games with Bandit Feedback.
In *Thirty-seventh Conference on Neural Information Processing Systems*, volume 36, 2023.

Constantinos Daskalakis and Ioannis Panageas.
The Limit Points of (Optimistic) Gradient Descent in Min-Max Optimization.
Advances in Neural Information Processing Systems, 31, 2018.

Arnab Maiti, Ross Boczar, Kevin Jamieson, and Lillian J Ratliff.
Query-Efficient Algorithms to Find the Unique Nash Equilibrium in a Two-Player Zero-Sum Matrix Game.
arXiv preprint arXiv:2310.16236, 2023.

Brendan O'Donoghue, Tor Lattimore, and Ian Osband.
Matrix Games with Bandit Feedback.
In *Uncertainty in Artificial Intelligence*, pages 279–289. PMLR, 2021.

David H Wolpert and William G Macready.
Coevolutionary Free Lunches.
IEEE Transactions on Evolutionary Computation, 9(6):721–735, December 2005.