Minimum Entropy Coupling with Bottleneck

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Introduction Consider the following Markov Chain modeling a general lossy compression framework:

Let us consider the following general lossy compression setup:

$$
X \xrightarrow{p_{T|X}} T \xrightarrow{q_{Y|T}} Y
$$

generate the code *T*. Subsequently, the probabilistic decoder qreconstructs *Y* from *T*. The objective

Instead of expected sample-wise distortion $\mathbb{E}[d(X, Y)]$, we use the logarithmic-loss $H(X|Y)$. $H(X|Y)$.

s.t. *X* \$ *T* \$ *Y,*

Decomposition Decomposition

 $M_{\rm H}$, the coupling with $M_{\rm H}$ and $M_{\rm H}$ and $M_{\rm H}$ and $M_{\rm H}$ and $M_{\rm H}$

3 generate the code *T*. Subsequently, the probabilistic decoder qreconstructs *Y* from *T*. The objective is to identify the encoder and decoder that minimize the distortion between *X* and *Y* , subject to a

Entropy-Bounded Info. Max. (EBIM)

Section 4.4.1 introduced a greedy search algorithm for identifying deterministic mappings with a

ministic mapping. This approach will enable us to bridge the gap between the mappings identified

guaranteed gap from the optimal. In this section, we identify optimal mapping optimal mapping close to any deter-

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Example. *Figure 4.3 depicts an example of optimal solutions in the neighborhood of a deterministic*

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Theorem 3 Theorem 3. *Let pXT defines a deterministic mapping T* = *g*(*X*)*, with I*(*X*; *T*) = *H*(*T*) = *Rg. We*

 $\left| \right.$ Around a deterministic mapping $T = g(x)$, defines transformations resulting in optimal Theorem 3. *Let pXT defines a deterministic mapping T* = *g*(*X*)*, with I*(*X*; *T*) = *H*(*T*) = *Rg. We l* solutions for small enduring $\frac{1}{2}$ solutions for small enduring $\frac{1}{2}$ solutions for small enduring $\frac{1}{2}$ $1.5.$ *I***EBIM** 2.6 *is achieved by moving and infinitesimal probability mass for the cell with mass for the cell with mass from the cell with mass* $\frac{1}{2}$

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\mathcal{I}_{EBIM}(p_X, R_g + \epsilon)
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2. $\mathcal{I}_{EBIM}(p_X, R_g - \epsilon)$

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b*, then*

b, we have:

Deterministic EBIM Solver

- The number of deterministic mappings in EBIM formulation is $O(n^n)$, where $n = |\mathcal{X}|$.
- Iterating over all deterministic mappings is not feasible.
- \bullet One should look for carefully constructed search algorithms to find such mappings with resulting $I(X; T)$ as close as possible to R.

Algorithm 5 Deterministic EBIM Solver 3. *^H* (*p ,* 1 *p*) = *p* log *p* (1 *p*) log(1 *p*) = *h*(*p*).

```
Input: p_X, ROutput: pXT
 1: p_{XT} \leftarrow diag(p_X)2: for i \leftarrow 1 to |p_X| - 1 do
 4: I_s^{(i)} \leftarrow Mutual Information imposed by p_s^{(i)}.
 5: p_{l}^{(i)} \leftarrow Merge the two columns with the largest sum in p_{XT}.
 6: I_l^{(i)} \leftarrow Mutual Information imposed by p_l^{(i)}.
 7: if I_s^{(i)} \n\t\leq R then<br>8: return p_s^{(i)}8: return p_s^{(i)}9: else if I_l^{(i)} \leq R < I_s^{(i)} then
10: return p_l^{(i)}11: else
12: p_{XT} \leftarrow p_l^{(i)}Proof. For the gap to the optimal objective, IEBIM(pX, R)  I
                Theorem 2. If the output of Algorithm 5 yields mutual information I
                whereaf information imposed by p_l.<br>
\leq R then \leq i \therefore p<sup>2</sup> d \leq p<sup>2</sup> d \leq p<sup>2</sup> d \leq i \leq i
                              . Equation (4.16) IEBIM(pX, R)  I
```
Theorem 2

3: $p_s^{(i)} \leftarrow$ Merge the two columns with the smallest sum in p_{XT} . information \hat{I} then. If the output of Algorithm 5 yields mutual \int information \tilde{I} , then:

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\mathcal{I}_{EBIM}(p_X,R) - I \leq h(p_2),
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b*, then*

where $h(\cdot)$ is the binary entropy function, and p_2 *p denotes the second largest element of* p_2 . where $h(\cdot)$ is the binary entropy function, and $p_i^{(i)} \leq R < I_s^{(i)}$ then
 $p_i^{(i)}$ are $p_i^{(i)}$
 p_2 denotes the second largest element of p_X .

Core Contributions

Minimum Entropy Coupling with Bottleneck

- We presents a lossy compression framework under logarithmic loss, that extends Min. Entropy Coupling (MEC) with a bottleneck.
- We then propose Entropy-Bounded Information Maximization (EBIM) formulation for the encoder, extensively characterize the structure of its optimal solution, and provide efficient approximate solutions with guaranteed performance.
- We also illustrate the practical application of MEC-B through experiments in Markov Coding Games under rate limits.