Semi-supervised learning: The provable benefits of unlabeled data for sparse Gaussian classification

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Limited theory for SSL (clustering assumption, manifold assumption, etc.)

SSL for sparse GMM

Observe L labeled samples $\{(\boldsymbol{x}_i, y_i)\}_{i=1}^{L}$ and n unlabeled $\{\boldsymbol{x}_i\}_{i=L+1}^{n+L}$ from the sparse Gaussian mixture model

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Intermediate task: accurately recover S

SL and UL for sparse GMM

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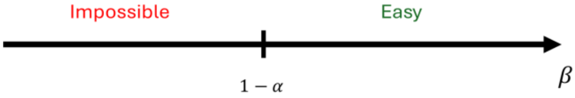

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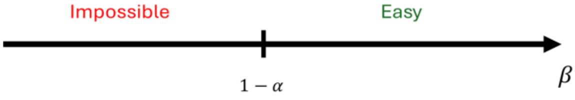
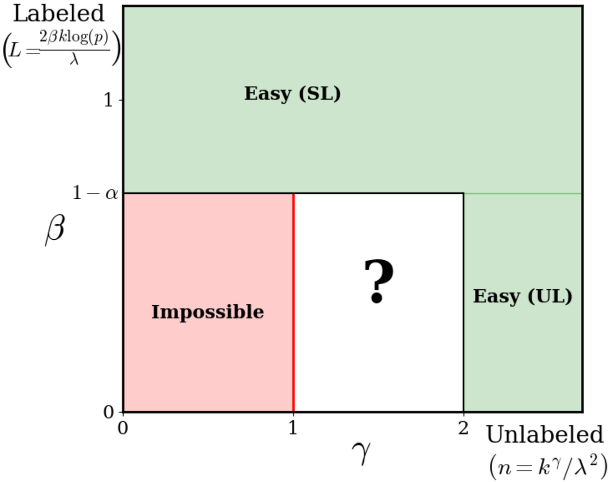
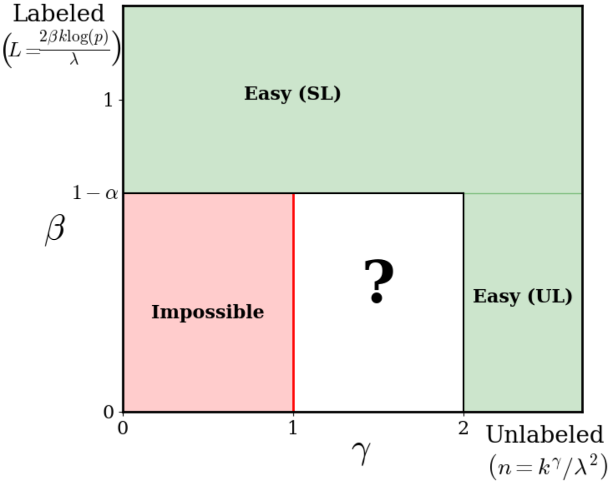


UL: Scaling $n = \frac{k^{\gamma}}{\lambda^2}$

[Verzelen and Arias-Castro 17' Fan et al 18, Loffler et al 22']

SSL phase space



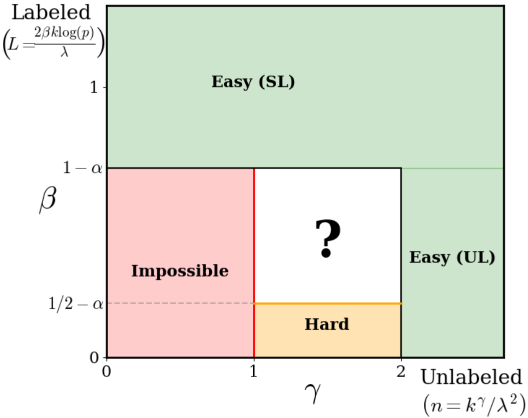


Computational Limits in SSL?

Computational lower bound

Theorem (SSL computational lower bound, informal)

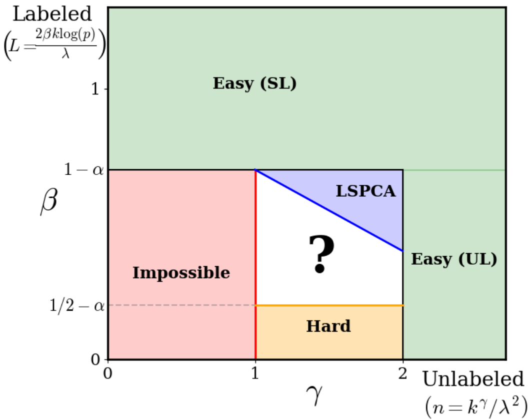
If $\beta < \frac{1}{2} - \alpha$ and $\gamma < 2$, then classification and variable selection are computationally hard.



Theoretical Guarantee for LSPCA

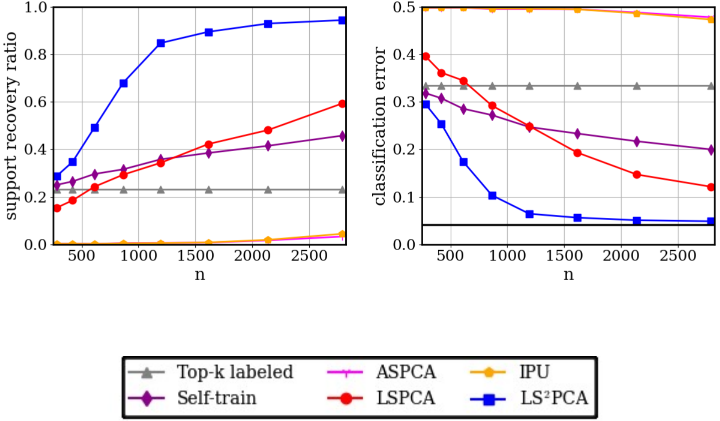
Theorem (informal)

If $\beta > 1 - \gamma\alpha$ then LSPCA with screening factor $\tilde{\beta} \in (1 - \gamma\alpha, \beta)$ recovers most of the support with high probability.



Simulation results: SSL vs. UL

$L = 200$ fixed, increase n



Simulation results: SSL vs. SL

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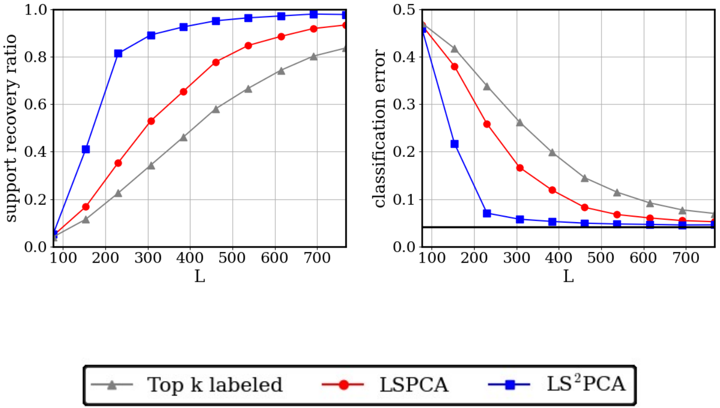

Thank You !