NEURAL INFORMATION PROCESSING SYSTEMS

Characteristic Circuits

Zhongjie Yu

TU Darmstadt
Germany

Martin Trapp

Aalto University
Finland

Kristian Kersting

TU Darmstadt/Hessian.AI/DFKI
Germany

Bundesministerium für Bildung und Forschung
Kompetenzzentren Arbeitsforschung
KompAKI
hessian.AI
DFKI
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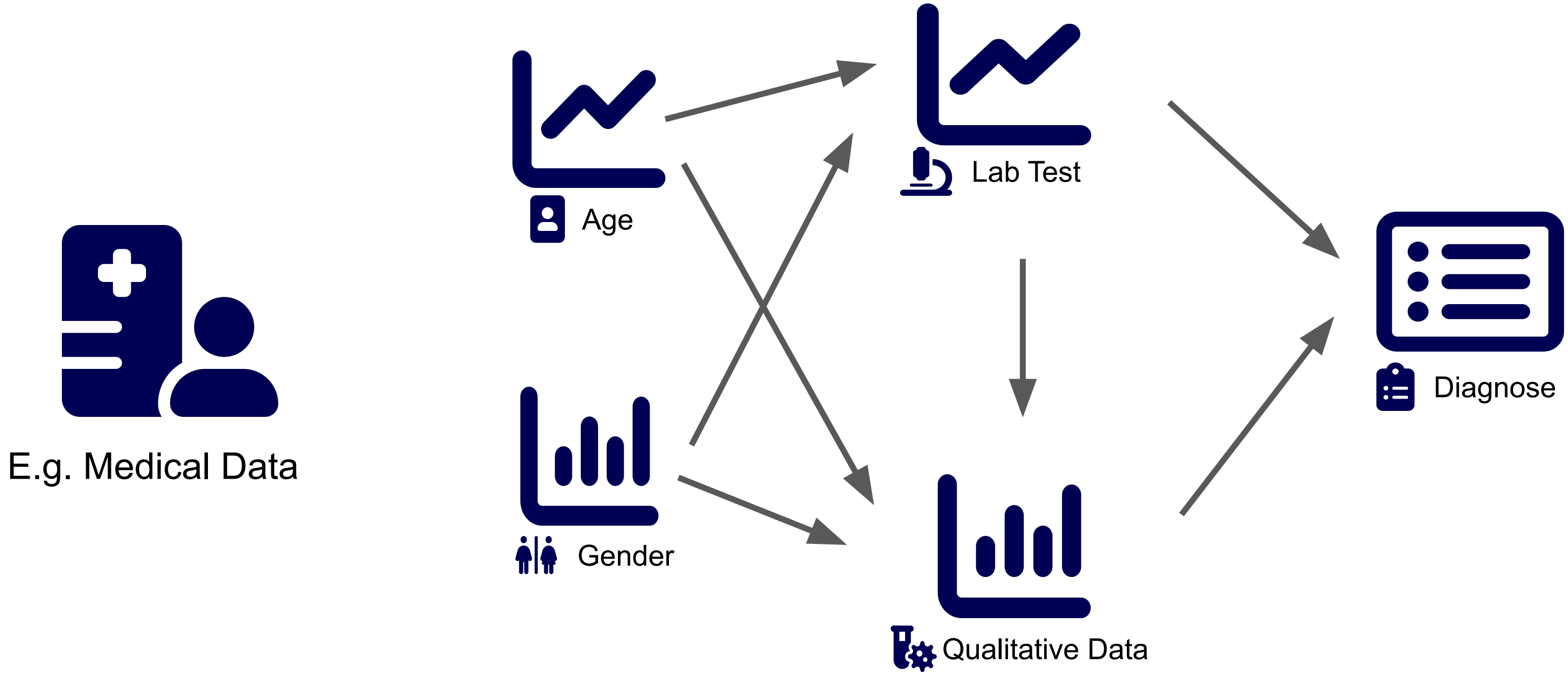
Real-world Data is often Hybrid



Probabilistic Circuits in a Nutshell

$$p(X_1)$$

[1] Figure from: Antonio Vergari et al., "*Tractable Probabilistic Models*", Talk at UAI Tutorial, 2019.
[2] YooJung Choi et al., "*Probabilistic circuits: A unifying framework for tractable probabilistic models*", Technical report, UCLA, 2020.

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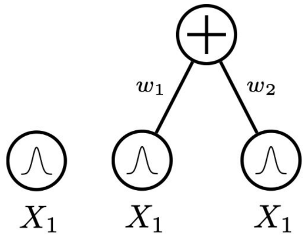

$p(X_1)$ $p(X_1)$

mixture

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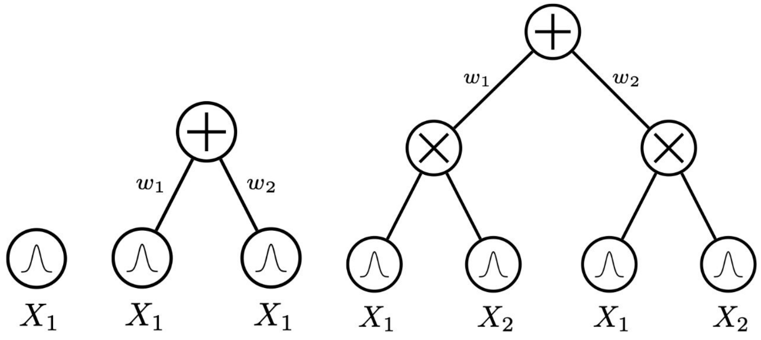

$p(X_1)$ $p(X_1)$ $p(X_1, X_2)$

mixture independence

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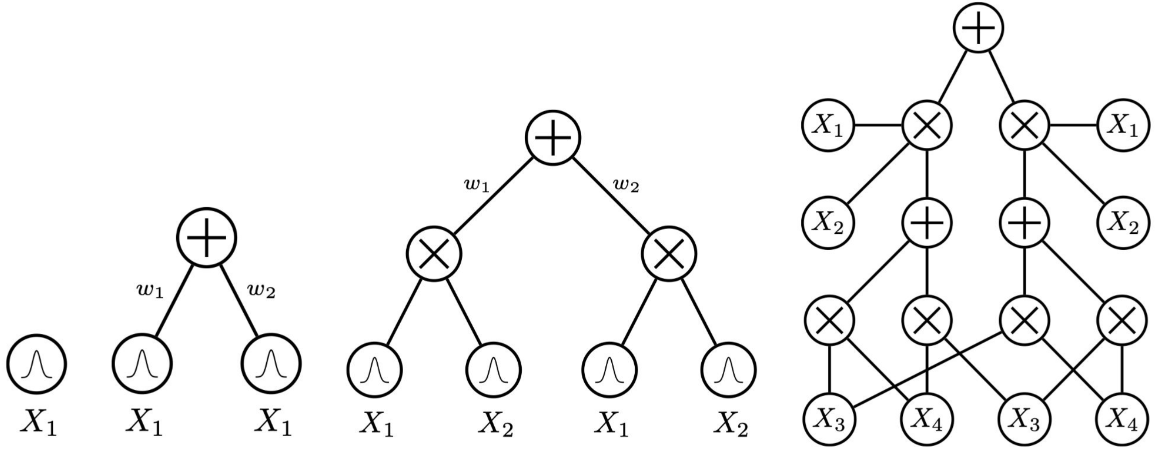

$p(X_1)$ $p(X_1)$ $p(X_1, X_2)$ $p(X_1, X_2, X_3, X_4)$

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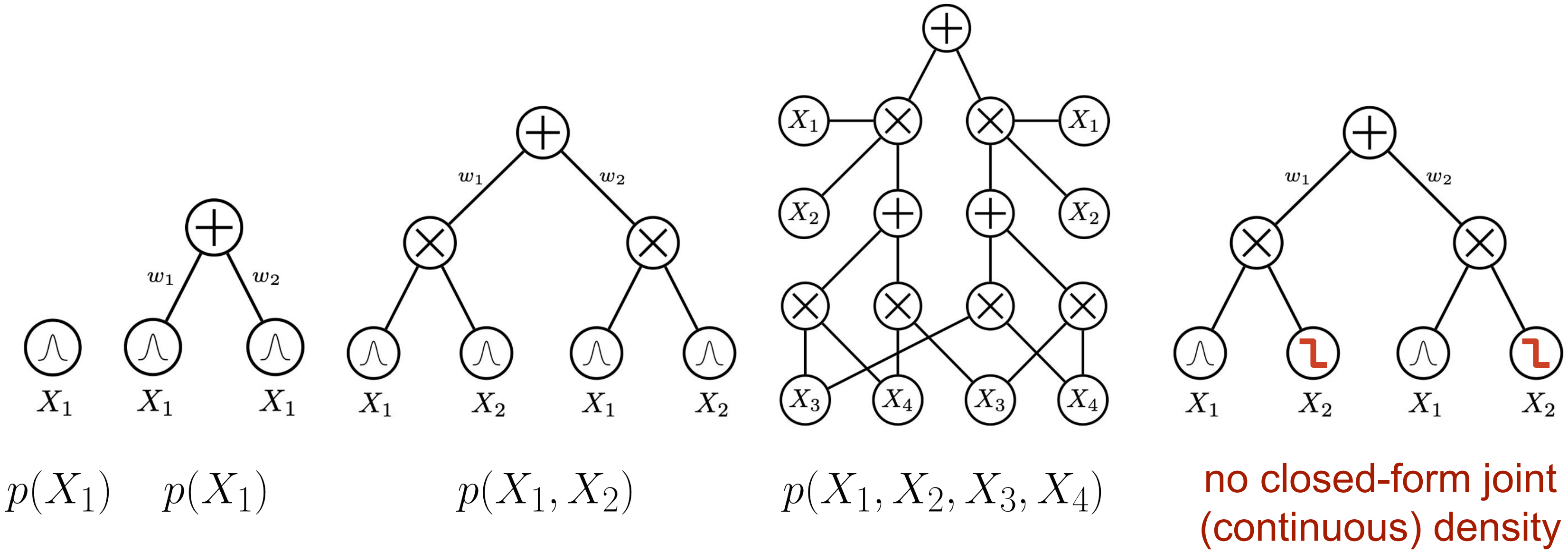

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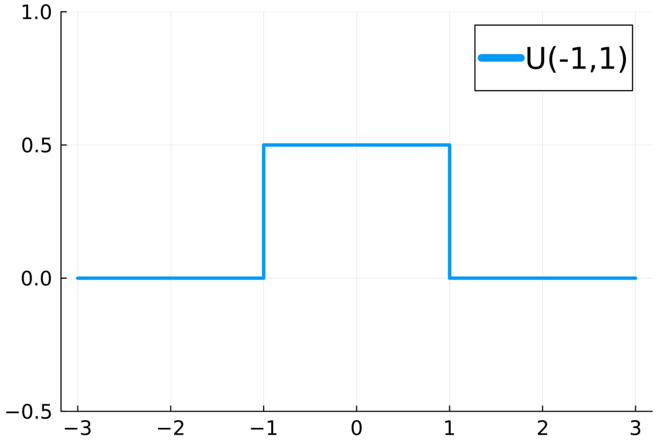
Characteristic Circuits in a Nutshell

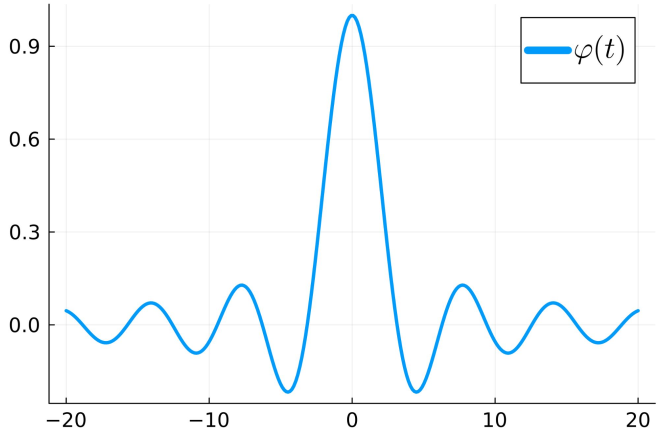
In mixed domains,
don’t go for histograms,
don’t go for PDFs,
don’t go for CDFs,
go for Characteristic Functions!

Characteristic Functions via the Fourier Transform

$$\varphi_{\boldsymbol{X}}(\boldsymbol{t}) = \mathbb{E}[\exp(\mathrm{i}\,\boldsymbol{t}^\top \boldsymbol{X})] = \int_{\boldsymbol{x}\in\mathbb{R}^d} \exp(\mathrm{i}\,\boldsymbol{t}^\top \boldsymbol{x})\,\mu_{\boldsymbol{X}}(\mathrm{d}\boldsymbol{x})$$

Uniform Distribution





[1] Zoltán Sasvári, “*Multivariate characteristic and correlation functions*”, volume 50. Walter de Gruyter, 2013.
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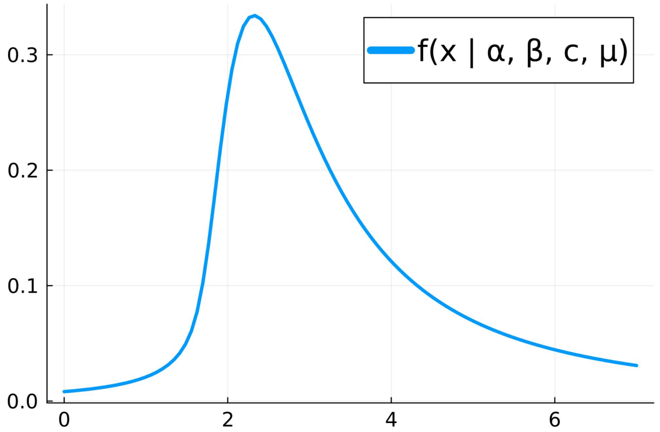
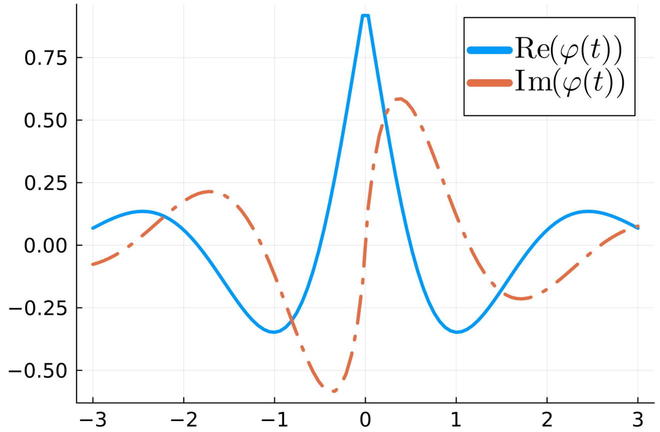
Alpha-Stable Distribution





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Categorical Distribution

$$P(X = j) = p_j \qquad \varphi_X(t) = \sum_{j=1}^{k} p_j \exp(\mathrm{i}\,t\,j)$$

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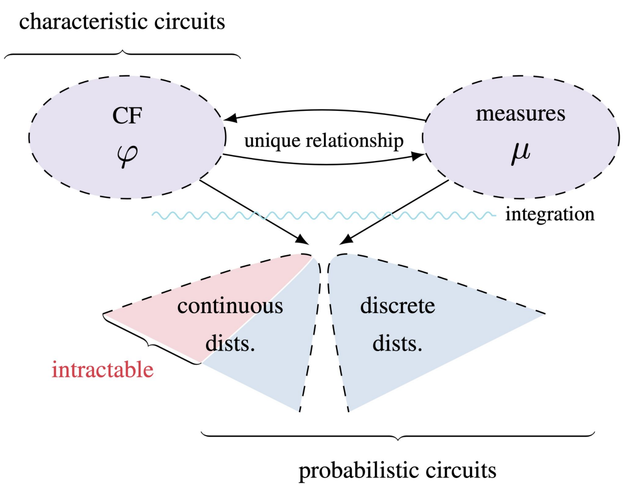
Characteristic Circuits are ...

- a tractable probabilistic model with
- a unified formalization of distributions over heterogeneous data
- in the continuous spectral domain with
- efficient probabilistic inference even when no closed-form density function is available



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Characteristic Circuit - A Recursive Definition

- a ***characteristic function*** for a scalar random variable is a *characteristic circuit*

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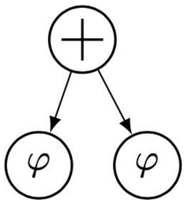
- a ***convex combination*** of characteristic circuits is a *characteristic circuit*

$$\int_{\boldsymbol{x}\in\mathbb{R}^d} \exp(\mathrm{i}\boldsymbol{t}^\top \boldsymbol{x}) \left[\sum\nolimits_{\mathsf{N}\in\mathrm{ch}(\mathsf{S})} w_{\mathsf{S},\mathsf{N}}\,\mu_{\mathsf{N}}(\mathrm{d}\boldsymbol{x})\right] = \sum\nolimits_{\mathsf{N}\in\mathrm{ch}(\mathsf{S})} w_{\mathsf{S},\mathsf{N}} \underbrace{\int_{\boldsymbol{x}\in\mathbb{R}^{p_{\mathsf{S}}}} \exp(\mathrm{i}\boldsymbol{t}^\top \boldsymbol{x})\,\mu_{\mathsf{N}}(\mathrm{d}\boldsymbol{x})}_{=\varphi_{\mathsf{N}}(\boldsymbol{t})}$$



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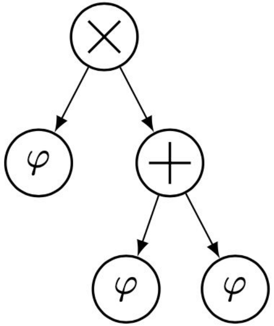

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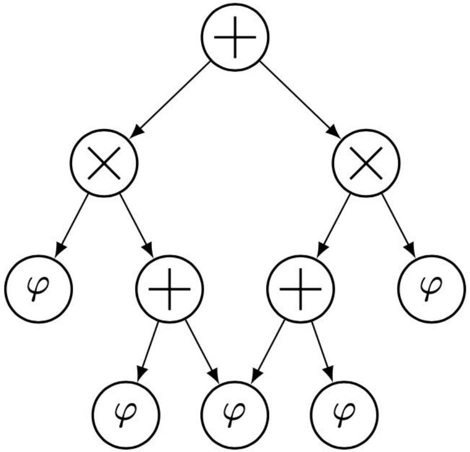
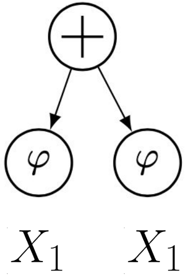
- a ***product*** of characteristic circuits is a *characteristic circuit*

$$\varphi_{\mathsf{P}}(\boldsymbol{t}) = \prod\nolimits_{\mathsf{N}\in\mathrm{ch}(\mathsf{P})} \varphi_{\mathsf{N}}(\boldsymbol{t}_{\psi(\mathsf{N})})$$

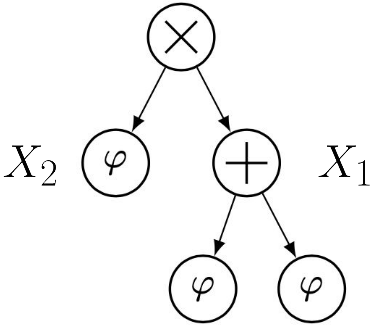
Characteristic Circuit - A Recursive Definition



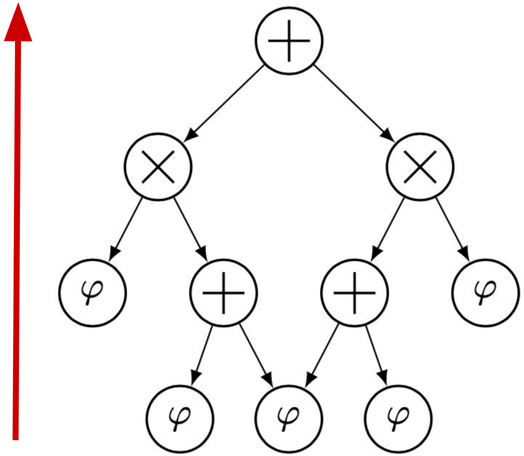
Smoothness:



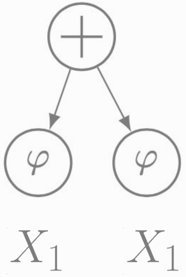
Decomposability:



Characteristic Circuit - A Recursive Definition



Smoothness:



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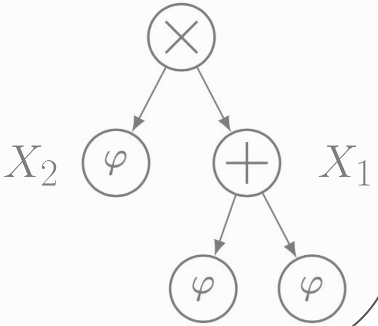
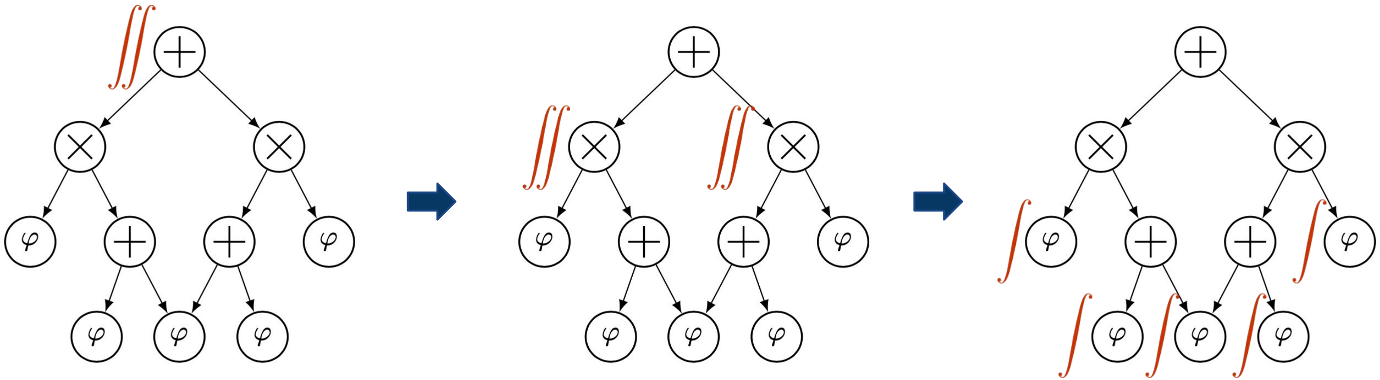
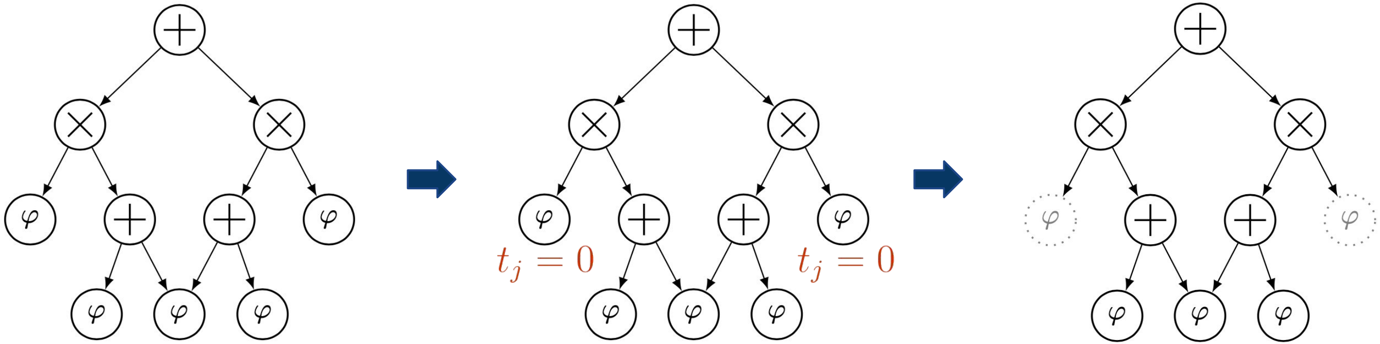


Efficient Computation of … Densities



Efficient computation through analytic or numerical integration at the leaves.

Efficient Computation of … Marginals



The marginal CC is obtained by setting the corresponding $t_j = 0$.

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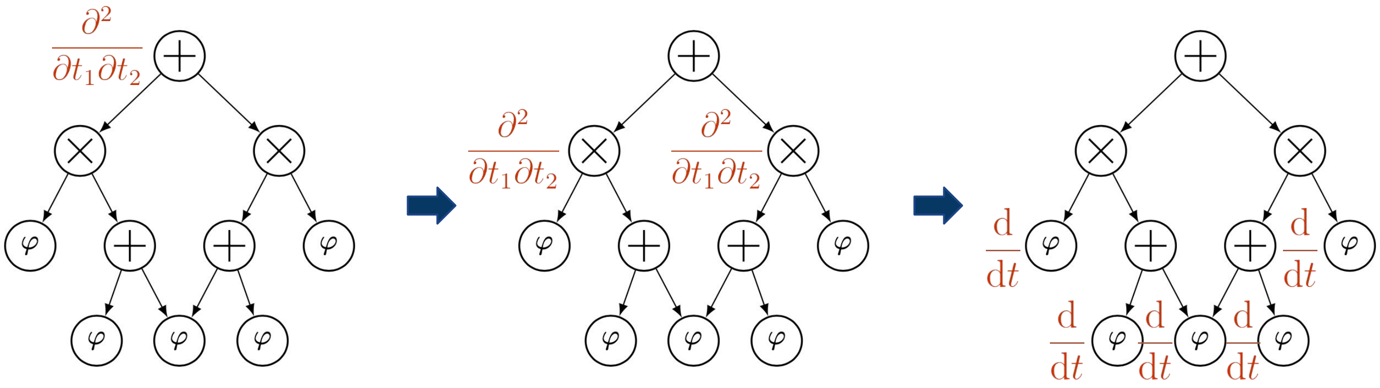
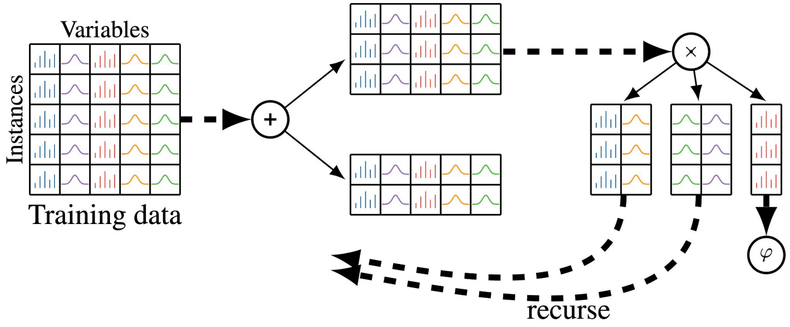
Efficient Computation of … Moments



The moments can be computed efficiently through differentiation of the circuit.

Learning Characteristic Circuits

- Structure Learning - inspired by the LearnSPN algorithm



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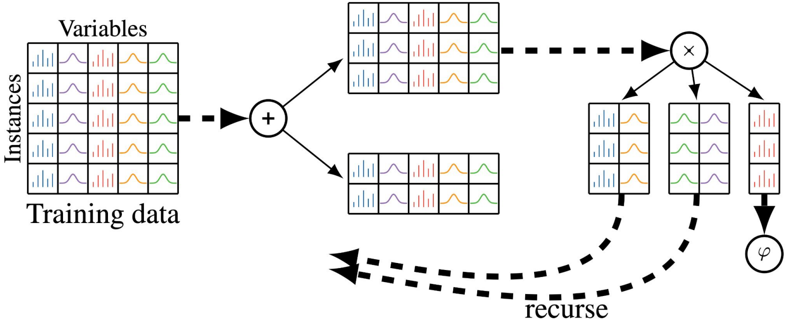


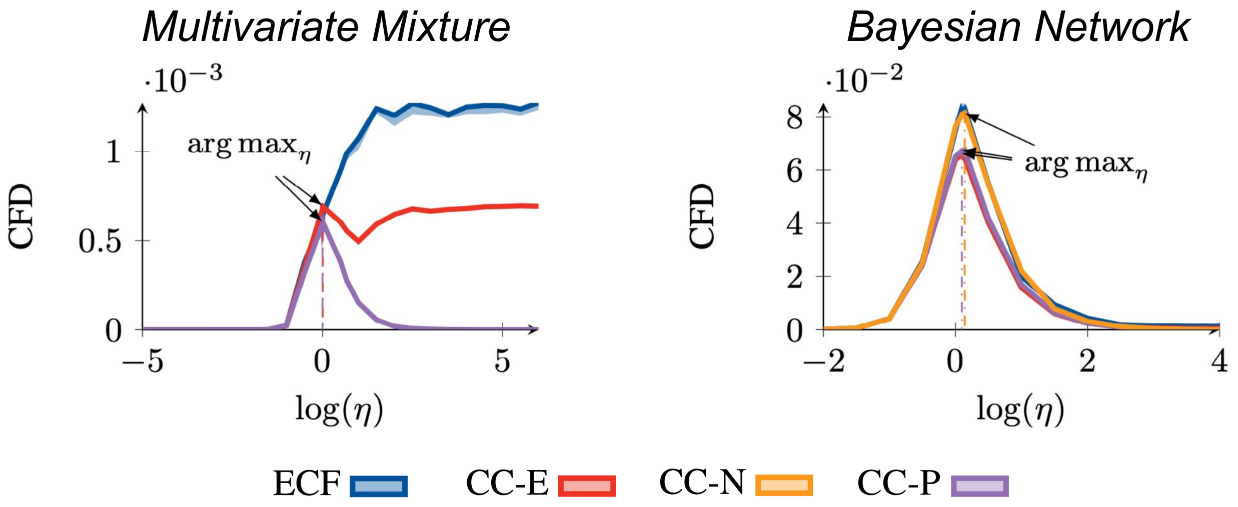
- Parameter Learning - minimizing the squared Characteristic Function Distance (CFD)

$$\frac{1}{k}\sum_{j=1}^{k}\left|\varphi_{\mathrm{ECF}}(\boldsymbol{t}_j) - \varphi_{\mathcal{C}}(\boldsymbol{t}_j)\right|^2 \quad \text{where} \quad \{t_1, \cdots, t_k\} \overset{\text{i.i.d.}}{\sim} \omega(\boldsymbol{t};\eta)$$

[1] Robert Gens and Pedro Domingos, "*Learning the structure of sum-product networks*", ICML, 2013.
[2] Abdul Fatir Ansari et al., "*A characteristic function approach to deep implicit generative modeling*", CVPR, 2020.

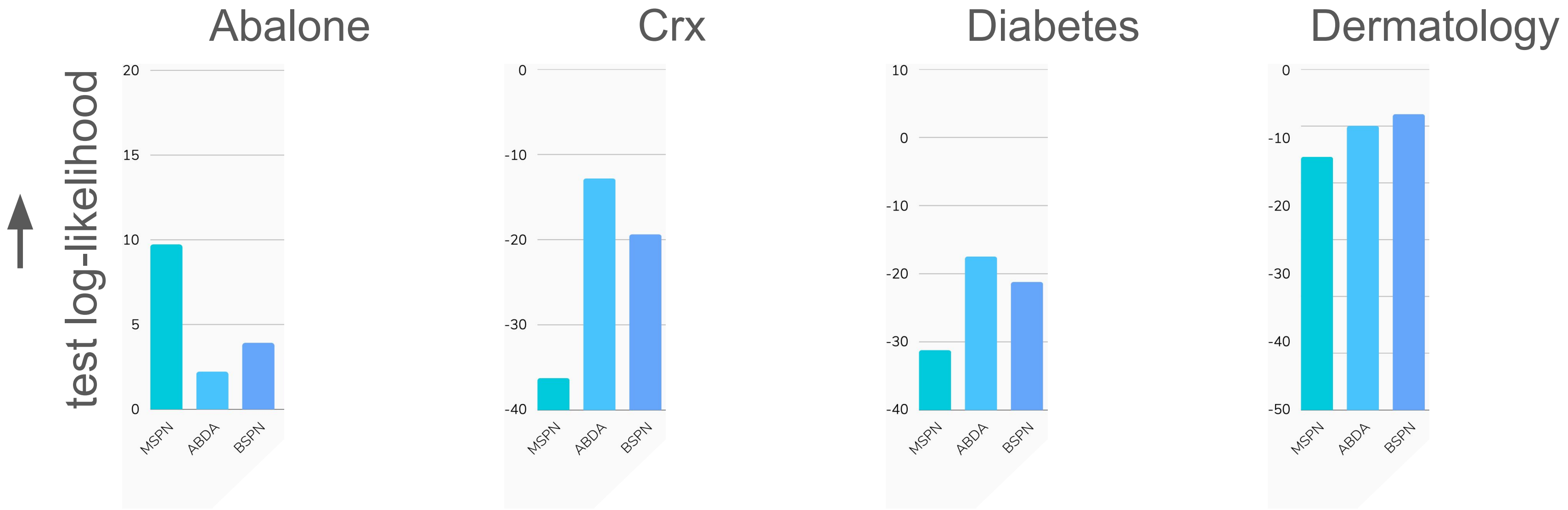
Characteristic Circuits Approximate Distributions Well

Compared to the widely used Empirical Characteristic Function (ECF, $\frac{1}{n}\sum_{j=1}^{n}\exp(\mathrm{i}\,t\,x_j)$)



Discrete RV	ECF	ECF	Gaussian	Categorical
Continuous RV		ECF	Gaussian	Gaussian

Better Density Estimators on Heterogeneous Data



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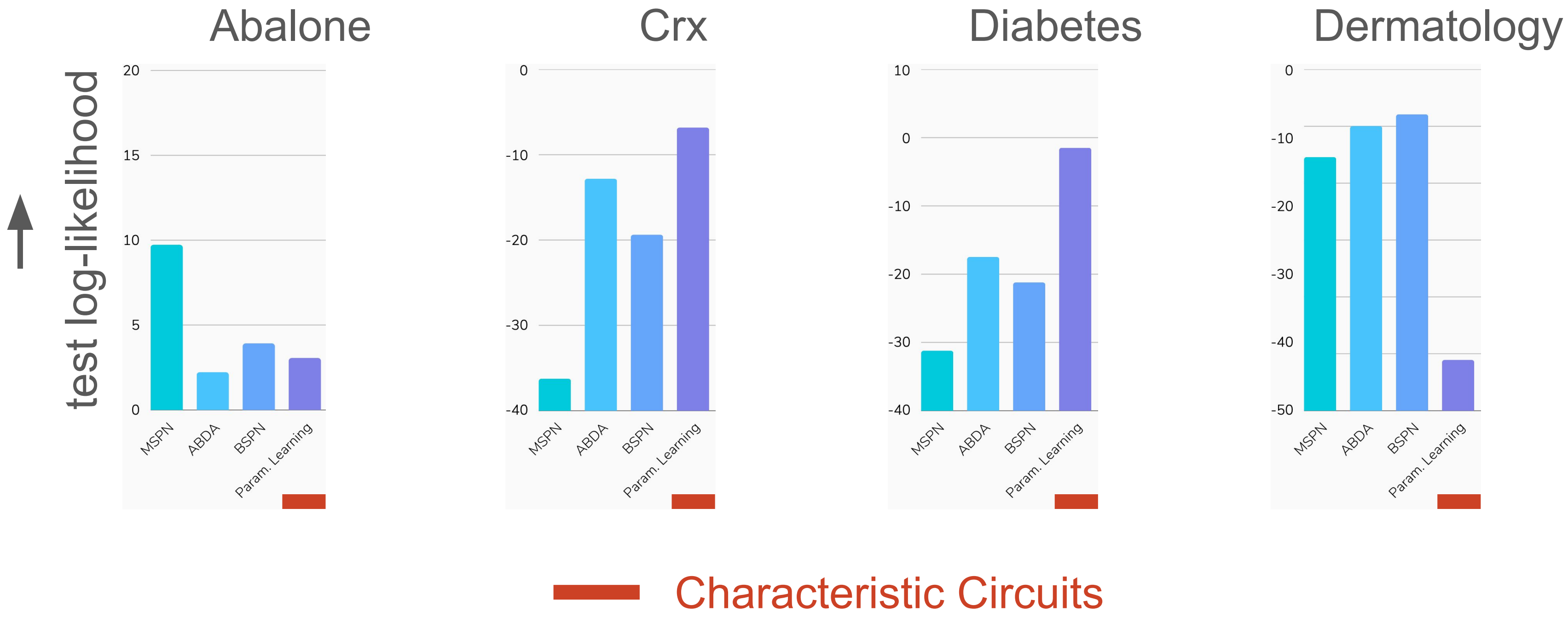

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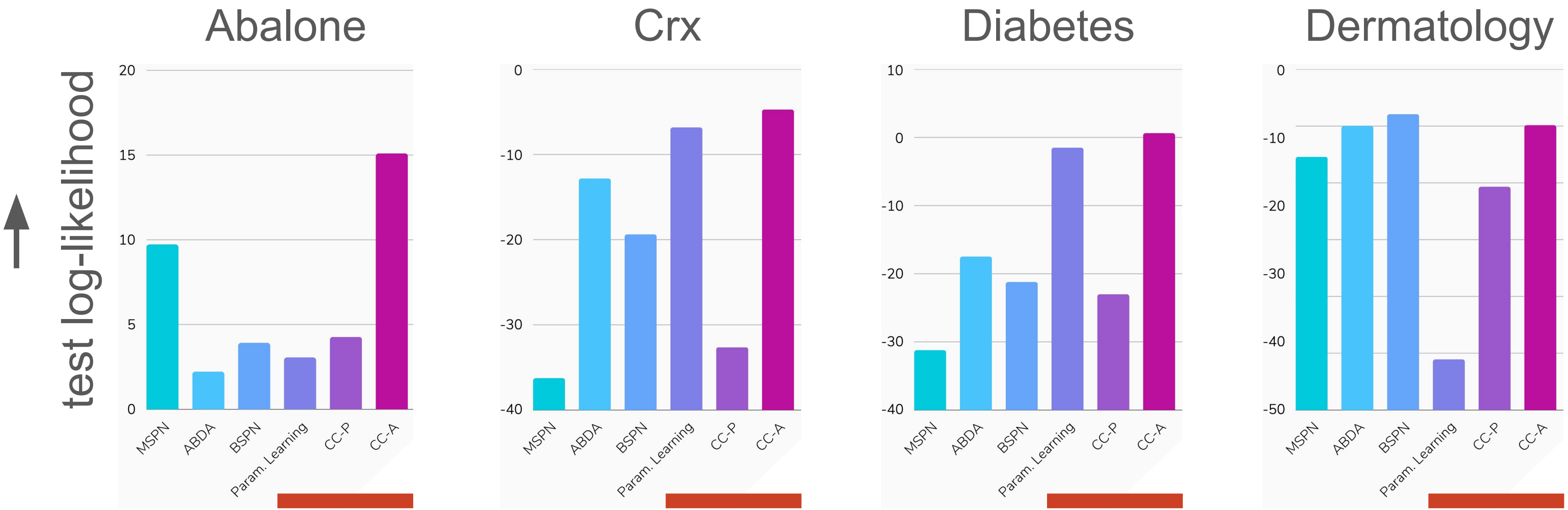

Characteristic Circuits

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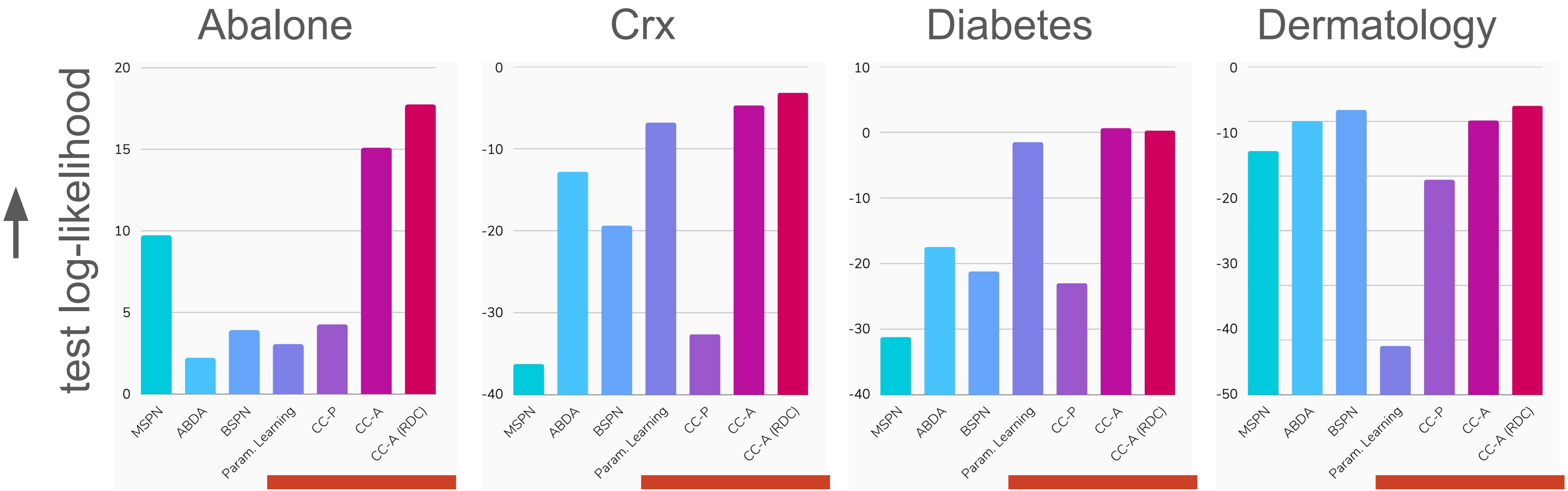

Characteristic Circuits

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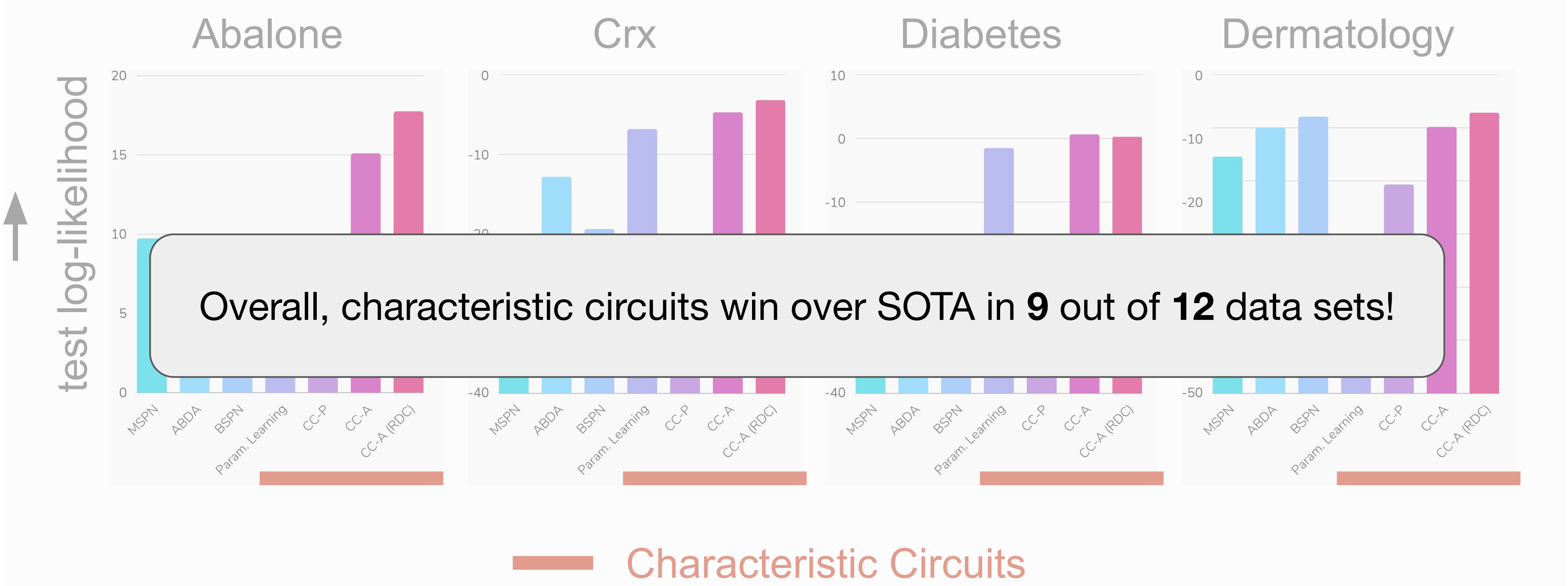

Overall, characteristic circuits win over SOTA in **9** out of **12** data sets!

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Take-Away

Characteristic Circuits:

- a *unifying view* of hybrid models in the *spectral domain*
- *tractable* densities, marginals, and moments
- *parameter* and *structure* learning

Future directions:

- other queries from CCs, e.g. sampling
- advanced structure for parameter learning
- Characteristic Flows
- …

Thank you!

Questions?

Paper

Code

NEURAL INFORMATION PROCESSING SYSTEMS

Should we use it?

- Characteristic Circuits are better density estimators on heterogeneous data.
  Characteristic Circuits provide higher LL on 9 out of 12 of the data sets.

Data Set	Parameter Learning	MSPN	ABDA	BSPN	Structure Learning		
					CC-P	CC-A	CC-A^{RDC}
Abalone	3.06	9.73	2.22	3.92	4.27	15.10	**17.75**
Adult	-14.47	-44.07	-5.91	-4.62	-31.37	-7.76	**-1.43**
Australian	-5.59	-36.14	-16.44	-21.51	-30.29	-3.26	**-2.94**
Autism	-27.80	-39.20	-27.93	**-0.47**	-34.71	-17.52	-15.5
Breast	-20.39	-28.01	-25.48	-25.02	-54.75	-13.41	**-12.36**
Chess	-13.33	-13.01	-12.30	**-11.54**	-13.04	-13.04	-12.40
Crx	-6.82	-36.26	-12.82	-19.38	-32.63	-4.72	**-3.19**
Dermatology	-45.54	-27.71	-24.98	-23.95	-30.34	-24.92	**-23.58**
Diabetes	-1.49	-31.22	-17.48	-21.21	-23.01	**0.63**	0.27
German	-19.54	-26.05	-25.83	-26.76	-27.29	-15.24	**-15.02**
Student	-33.13	-30.18	-28.73	-29.51	-31.59	-27.92	**-26.99**
Wine	0.32	-0.13	-10.12	-8.62	-6.92	13.34	**13.36**
# best	0	0	0	2	0	1	9

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- Characteristic Circuits are better density estimators on heterogeneous data.
  Characteristic Circuits provide higher LL on 9 out of 12 of the data sets.

Data Set	Parameter Learning	MSPN	ABDA	BSPN	Structure Learning		
					CC-P	CC-A	CC-A^{RDC}
Abalone	3.06	9.73	2.22	3.92	4.27	15.10	**17.75**
Adult	-14.47	-44.07	-5.91	-4.62	-31.37	-7.76	**-1.43**
Australian	-5.59	-36.14	-16.44	-21.51	-30.29	-3.26	**-2.94**
Autism	-27.80	-39.20	-27.93	**-0.47**	-34.71	-17.52	-15.5
Breast	-20.39	-28.01	-25.48	-25.02	-54.75	-13.41	**-12.36**
Chess	-13.33	-13.01	-12.30	**-11.54**	-13.04	-13.04	-12.40
Crx	-6.82	-36.26	-12.82	-19.38	-32.63	-4.72	**-3.19**
Dermatology	-45.54	-27.71	-24.98	-23.95	-30.34	-24.92	**-23.58**
Diabetes	-1.49	-31.22	-17.48	-21.21	-23.01	**0.63**	0.27
German	-19.54	-26.05	-25.83	-26.76	-27.29	-15.24	**-15.02**
Student	-33.13	-30.18	-28.73	-29.51	-31.59	-27.92	**-26.99**
Wine	0.32	-0.13	-10.12	-8.62	-6.92	13.34	**13.36**
# best	0	0	0	2	0	1	9

[1] Alejandro Molina et al., "*Mixed sum-product networks: A deep architecture for hybrid domains*", AAAI, 2018.
[2] Antonio Vergari et al., "*Automatic bayesian density analysis*", AAAI, 2019.
[3] Martin Trapp et al., "*Bayesian learning of sum-product networks*", NeurIPS, 2019.