Koopman Kernel Regression

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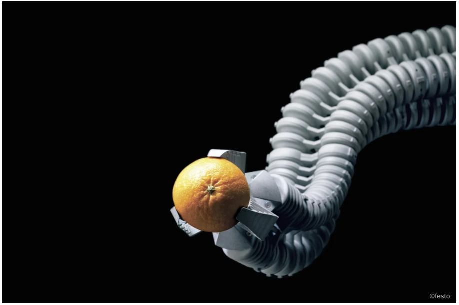
37th Conference on Neural Information Processing Systems (NeurIPS 2023)

Simple yet Expressive Representations of Complex Dynamics



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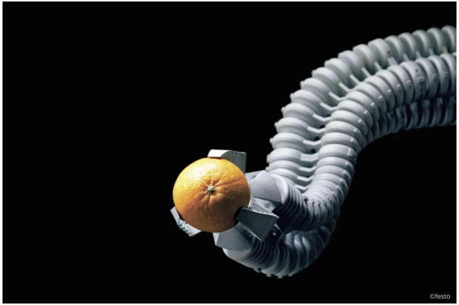

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simple & universal **representations**

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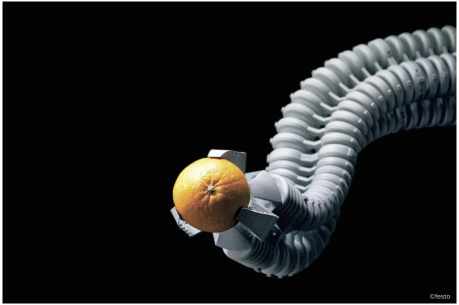

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simple & universal **representations**
guaranteed **generalization** & **consistency**

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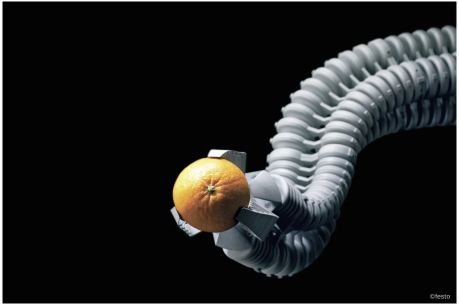

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) *facilitating accelerated optimization-based decision making*

Take the best of time-series, ODEs and RKHS

Time-series decomposition

discriminative3
linear 3
time-variant 7

ODEs

generative 3
nonlinear7
time-invariant 3

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Learning in the best of both worlds?

responses2 span(time-invariant behaviors) 2 RKHS

Linear & Dynamics-invariant RKHS for Learning Dynamics

Koopmanism

Symbol K: the time-shift or Koopman operator, so $K y(k) = y(k+1)$.

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Koopman eigenfunctions

$$K^{k} \; {}_{j} = \; {}^{k}_{j} \; {}_{j}$$

! evolve in time with amplitude $j \; {}_{j} j$ & frequency $\arg(\; {}_{j})$

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Ours

invariant RKHSs fH $_j$ g

Existing

[Kostic+ 2022; Klus+ 2020]

Koopman Kernel Regression

Koopman kernel (ridge) regression

Given N i.i.d. state-output trajectories $\{x^{(i)}, y^{(i)}\}_{i=1}^{N} \in (X \times Y)^N$ of length H, compute

Koopman kernel

$$\hat{M} = \arg\min_{M \in S(H_1 \times \cdots \times H_D)} \left(\frac{1}{N} \sum_{i \in [N]} \| y^{(i)} - M(x^{(i)}) \|_Y^2 + \lambda \| M \|_H^2 \right) \in \operatorname{span}\{ K(\cdot, x^{(i)}) \}$$

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minimizes multi-step empirical risk

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approximation error vanishing with rank D

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rank-independent generalization

Practical Implications

Flow past a cylinder

Forecasting velocity magnitude of sensor from 50 100-dimensional initial conditions.

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Flow past a cylinder

Forecasting velocity magnitude of sensor from 50 100-dimensional initial conditions.

Orders-of-magnitude greater accuracy (also for a wide range of hyperparameters)

Also in the Full Paper...

Including but not limited to

- full theoretical results
- numerical validation
- extensive comparisons
- complexity analysis
- forecasting and training times

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Spoiler: superior to *Koopman operator regression* across the board

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