Online Algorithms for the Santa Claus Problem

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NEURAL INFORMATION PROCESSING SYSTEMS

Background

Our work focuses on the problem of **online resource allocation** where we provide the first near-optimal algorithm for the Santa Claus problem in an online setting.

Such an algorithm is sorely lacking in the literature and has widespread practical applications outside of the theoretical community, such as:

- Allocating ad slots to advertisers
- Allocating compute resources to users
- Allocating blood donors to blood banks
- Allocating food donations to food banks

We provide nearly tight results for the **adversarial** and **random order** problem instances.

Problem Definition

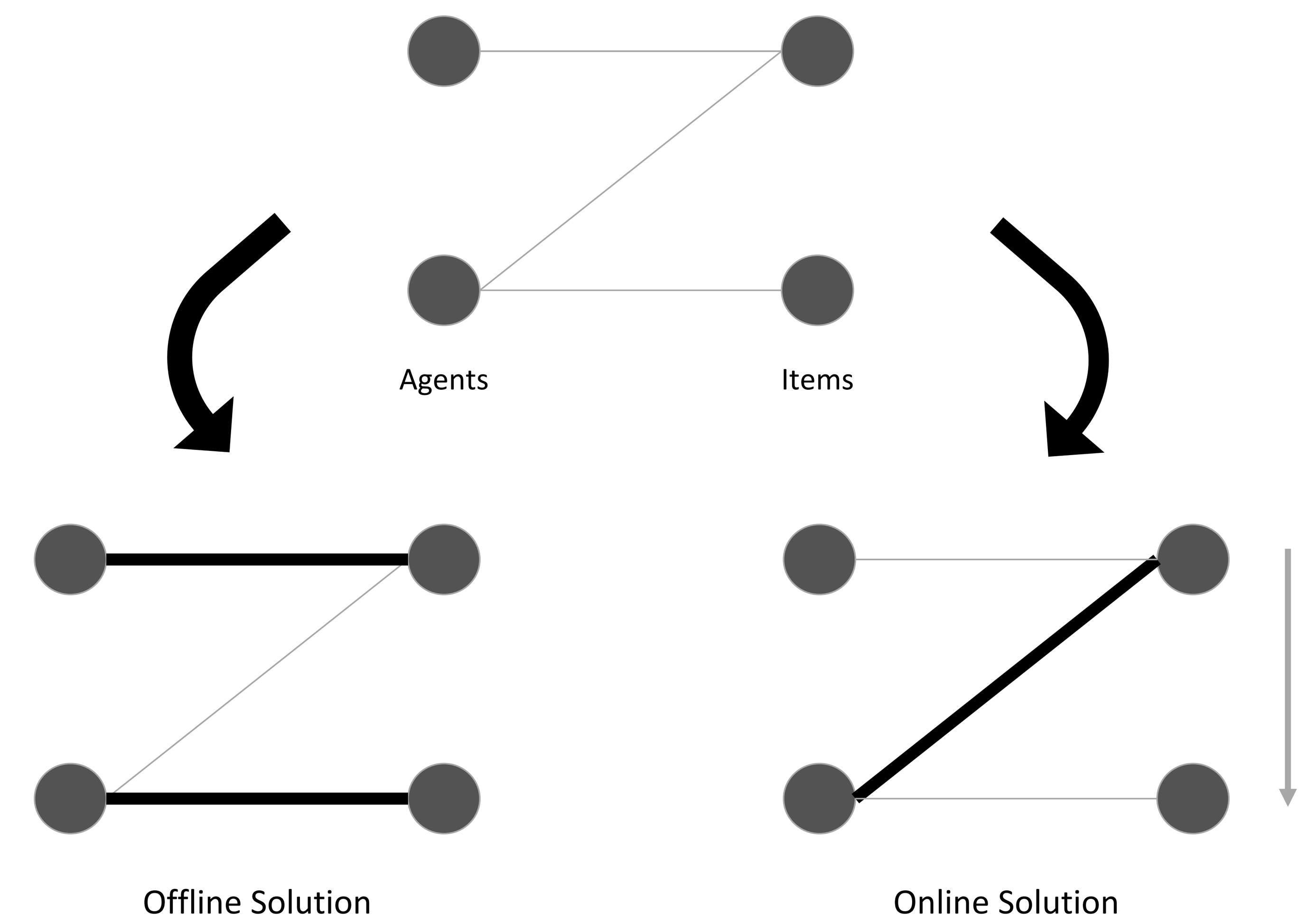
We can informally define the problem as follows:

Santa Claus has a set of m presents to be distributed equitably among n children. Each child $i \in [n]$ has some arbitrary non-negative value v_{ij} for present $j \in [m]$. Santa's goal is to distribute the presents in a way that makes the least satisfied child maximally satisfied. More formally, this means that the assignment seeks to maximize the minimum total value of the presents received by any child, where the total value of presents received by a child is the sum of her values for the presents that she received. The Santa Claus problem can be formalized as the following integer program:

$$\max\left\{\min \sum_{j=1}^{m} v_{ij}x_{ij} \mid \sum_{i=1}^{n} x_{ij} \leq 1 \ \forall j \in [m], x \in \{0,1\}^{mn}\right\}$$

There is substantial literature going back more than 50 years that studies variants of this problem in the offline setting (see related work). In many practical situations, however, the set of items to be allocated is not known in advance. Motivated by this and the numerous applications of such allocation problems, we explore the Santa Claus problem in the **online setting**.

Offline and Online Settings



In the **offline** problem setting, the algorithm has access to each item j's value to every agent i and is tasked with allocating the items in a fair manner subject to the max-min constraint.

The **online** setting limits an algorithms knowledge by only presenting the valuation of each item *as it arrives* in an online input stream. The algorithm must make an irrevocable allocation decision at this arrival time before seeing subsequent items.

Main Algorithm

ALGORITHM 1: SMOOTH GREEDY WITH RESTART

Input: $0 < \varepsilon < 1$, input stream $\mathcal{M}$ of m items
Define $\phi_\varepsilon(u) = -\frac{1}{\varepsilon}\ln\left(\sum_i e^{-\varepsilon u_i}\right)$;
for $t = 1$ *to* $m/2$ **do**
  Select $\mathbf{x}^t \in \Delta^n$ to maximize $\phi_\varepsilon\left(\sum_{\tau=1}^{t} \mathbf{V}^\tau\right)$;
end
for $t = m/2+1$ *to* m **do**
  Select $\mathbf{x}^t \in \Delta^n$ to maximize $\phi_\varepsilon\left(\sum_{\tau=\frac{m}{2}+1}^{t} \mathbf{V}^\tau\right)$;
end

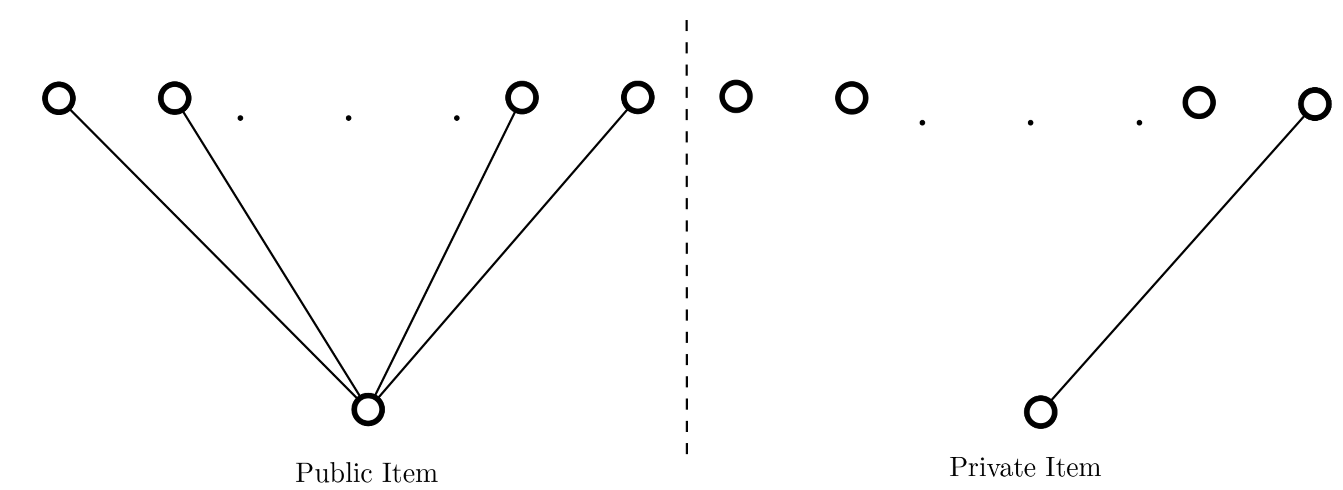
Main Results

We here present our main results:

Theorem 1. In the adversarial input setting, no algorithm can obtain a competitive ratio better than $1/n$.

Theorem 2. For any $\varepsilon > 0$, there is an online fractional algorithm for the Santa Claus problem that has a competitive ratio of $(1-\varepsilon)$ in the random order input model under the assumption that $OPT \geq \Omega(\log n/\varepsilon^2)$.

Theorem 3. For any $\varepsilon \in (0,1)$, there is no online algorithm for the Santa Claus problem in the random order input model that has a competitive ratio of $(1-\varepsilon)$ when $OPT < C \cdot \ln n/\varepsilon$ for some (absolute) constant $C > 0$.



Theorems 1 and 3 are derived from hardness instances that are constructed using the above "types" of items to be allocated across agents, wherein the number of public and private items dictate the optimal solution size.

Our main algorithmic guarantee of Theorem 2 is achieved by the "Smooth Greedy with Restart" algorithm presented on the left. The near optimal result in the random order setting leverages this symmetry in preventing under saturation of any one agent in the system.

For full details on these results and a comprehensive review of the literature on online allocation, we invite you to read the full paper, linked here via QR code.

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