Counting the Optimal Solutions in Graphical Models

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Motivation and Contribution

- Combinatorial optimization in graphical models
  - Solution that optimizes a global objective function

$$x^* = \text{argmin}_x \sum_\alpha f_\alpha(x_\alpha)$$

- **NP-hard**: exponentially many terms
- **#opt: count the optimal solutions**
  - Naive brute-force approaches based on enumeration
    - Infeasible in practice if many optimal solutions
  - Introduce efficient variable elimination and search based algorithms that do not rely on enumeration

#opt

- Formally: $|S|, \; S = \{x \mid f(x) = V^*, V^* = \min_x \sum_\alpha f_\alpha(x_\alpha)\}$
  - Computed efficiently, without enumeration

- The #opt semiring: $\mathcal{A} = \langle \mathbb{R}^2, \otimes, \oplus \rangle \qquad (v, c) \in \mathbb{R}^2$

value

count

$$(v_1, c_1) \otimes (v_2, c_2) = (v_1 + v_2, c_1 \cdot c_2)$$

$$(v_1, c_1) \oplus (v_2, c_2) = \begin{cases} (v_1, c_1 + c_2), & \text{if } v_1 = v_2 \\ (v_1, c_1), & \text{if } v_1 < v_2 \\ (v_2, c_2), & \text{if } v_1 > v_2 \end{cases}$$

$\otimes$ - combination operator $\qquad \oplus$ - marginalization operator

Property: $\otimes$ distributes over $\oplus$

Exact Algorithms for #opt

- **Variable Elimination (VE)**
  - Eliminate variables following an ordering
  - Local computations facilitated by the distributivity property of the semiring
  - **Complexity**: O(n exp(w*)) w* - treewidth
- **AND/OR Branch-and-Bound Search (AOBB)**
  - Explore the context-minimal AND/OR search graph
  - Heuristic evaluation function to prune unpromising regions of the search space
  - **Complexity**: O(n exp(w*)) w* - treewidth
    - In practice, more efficient due to pruning