

Counting the Optimal Solutions in Graphical Models

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Motivation and Contribution

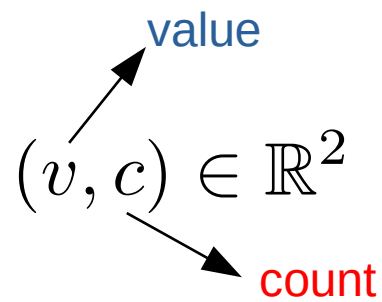
- Combinatorial optimization in graphical models
 - Solution that optimizes a global objective function

$$x^* = \operatorname{argmin}_x \sum_{\alpha} f_{\alpha}(x_{\alpha})$$

- **NP-hard**: exponentially many terms
- **#opt: count the optimal solutions**
 - Naive brute-force approaches based on enumeration
 - Infeasible in practice if many optimal solutions
 - Introduce efficient variable elimination and search based algorithms that do not rely on enumeration

#opt

- **Formally:** $|S|$, $S = \{x \mid f(x) = V^*, V^* = \min_x \sum_{\alpha} f_{\alpha}(x_{\alpha})\}$
 - Computed efficiently, without enumeration

- **The #opt semiring:** $\mathcal{A} = \langle \mathbb{R}^2, \otimes, \oplus \rangle$ $(v, c) \in \mathbb{R}^2$


$$(v_1, c_1) \otimes (v_2, c_2) = (v_1 + v_2, c_1 \cdot c_2)$$

$$(v_1, c_1) \oplus (v_2, c_2) = \begin{cases} (v_1, c_1 + c_2), & \text{if } v_1 = v_2 \\ (v_1, c_1), & \text{if } v_1 < v_2 \\ (v_2, c_2), & \text{if } v_1 > v_2 \end{cases}$$

\otimes - combination operator

\oplus - marginalization operator

Property: \otimes distributes over \oplus

Exact Algorithms for #opt

- **Variable Elimination (VE)**
 - Eliminate variables following an ordering
 - Local computations facilitated by the distributivity property of the semiring
 - **Complexity:** $O(n \exp(w^*))$ w^* - treewidth
- **AND/OR Branch-and-Bound Search (AOBB)**
 - Explore the context-minimal AND/OR search graph
 - Heuristic evaluation function to prune unpromising regions of the search space
 - **Complexity:** $O(n \exp(w^*))$ w^* - treewidth
 - In practice, more efficient due to pruning