(Probably) Concave Graph Matching

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Graph Matching

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\min_{X \in \Pi_n} -\text{tr}(AXBX^T)
\]
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Previous Work

• Superiority of the indefinite relaxation
  • [Lyzinski et al. PAMI 2016]

• Efficient graph matching via concave energies
  • [Vestner et al. CVPR 2017, Boyarski et al. 3DV 2017]
Advantages of Concave Relaxations

- All local minima are permutation matrices
Many important graph matching problems are concave!
Which $A, B$ give rise to concave relaxations?
Concavity of Indefinite Relaxation

• **Theorem:** It is sufficient that

\[ A = \Phi(x_i - x_j), \quad B = \Psi(y_i - y_j) \]

where \( \Phi, \Psi \) are positive definite functions of order one.
Concavity of Indefinite Relaxation

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Concave Energies

Euclidean distance in any dimension
• Mahalanobis distances
• Spectral graph distances
• Matching objects with deep descriptors

\[ A_{ij} = \| x_i - x_j \|_2 \]

Spherical distance in any dimension
[bogomolny, 2007]

\[ A_{ij} = d_{S^n}(x_i, x_j) \]
Do we really need a concave relaxation?
Do we really need a concave relaxation?

Image taken from Crane et al. 2017
• Theorem (upper bound on the probability of convex restriction)
Let $M \in \mathbb{R}^{m \times m}$ and $D \leq \mathbb{R}^m$ a uniformly sampled $d$-dimensional subspace, then:

$$Pr \left[ M \bigg| D > 0 \right] \leq \min_t \prod_{i=1}^{n} (1 - 2t\lambda_i)^{-d/2}$$
• Theorem (upper bound on the probability of convex restriction)

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$$
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$$

\[
\operatorname{diag}\left(\underbrace{-1, -1, \ldots, -1}_{0.51m}, \underbrace{1, 1, \ldots, 1}_{0.49m}\right)
\]
Applications
Conclusion

• A large family of concave or probably concave relaxations
• Checking probable concavity with eigenvalue bound
• Extension of [Lyzinsky et al. 2016] to practical matching problems
The End

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- Thanks for listening!