Direct Runge-Kutta Discretization Achieves Acceleration

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Acceleration in first order convex optimization

Optimize smooth convex function:
\[
\min_{x \in \mathbb{R}^d} f(x)
\]
Acceleration in first order convex optimization

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$$\min_{x \in \mathbb{R}^d} f(x)$$

Gradient Descent:

$$x_{k+1} = x_k - \eta \nabla f(x_k)$$
Acceleration in first order convex optimization

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\min_{x \in \mathbb{R}^d} f(x)
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Gradient Descent:

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x_{k+1} = x_k - \eta \nabla f(x_k)
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\[
\eta \to 0
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\[
\dot{x} = -\nabla f(x)
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Acceleration in first order convex optimization

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Gradient Descent:
\[
x_{k+1} = x_k - \eta \nabla f(x_k)
\]

As \(\eta \to 0\),
\[
\dot{x} = -\nabla f(x)
\]

\[
f(x(t)) - f(x^*) = \mathcal{O}\left(\frac{1}{t}\right)
\]
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\[
\eta \rightarrow 0
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f(x(t)) - f(x^*) = \mathcal{O}\left(\frac{1}{t}\right)
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Accelerated Gradient Descent [Nesterov 1983]:

\[
x_{k+1} = y_k - \eta \nabla f(y_k)
\]

\[
y_{k+1} = x_{k+1} + \beta (x_{k+1} - x_k)
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Acceleration in first order convex optimization

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Gradient Descent:
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\[ \eta \to 0 \]
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\[ x_{k+1} = y_k - \eta \nabla f(y_k) \]
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\[ \ddot{x} + \frac{3}{t} \dot{x} + \nabla f(x) = 0 \]

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Convergence in continuous time

\[ \ddot{x} + \frac{3}{t} \dot{x} + \nabla f(x) = 0 \quad f(x(t)) - f(x^*) = O\left(\frac{1}{t^2}\right) \]
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\[ t \to t^{p/2} \]

Arbitrary acceleration by change of variable

\[ \ddot{x} + \frac{2p + 1}{t} \dot{x} + C p^2 t^{p-2} \nabla f(x) = 0 \quad f(x(t)) - f(x^*) = \mathcal{O}\left(\frac{1}{t^p}\right) \]

Convergence in continuous time

\[ \ddot{x} + \frac{3}{t} \dot{x} + \nabla f(x) = 0 \tag{1} \]

\[ f(x(t)) - f(x^*) = \mathcal{O}\left(\frac{1}{t^2}\right) \tag{2} \]

\[ t \rightarrow t^{p/2} \]

Arbitrary acceleration by change of variable \[ [\text{WWJ 2016}] \]

\[ \ddot{x} + \frac{2p + 1}{t} \dot{x} + C_p t^{p-2} \nabla f(x) = 0 \tag{3} \]

\[ f(x(t)) - f(x^*) = \mathcal{O}\left(\frac{1}{t^p}\right) \tag{4} \]

However, smooth convex optimization algorithms cannot achieve faster rate than: \[ \mathcal{O}\left(\frac{1}{t^2}\right) \]

Question: How to relate the convergence rate in continuous time ODE to the convergence rate of a discrete optimization algorithm?
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Our approach: Discretize the ODE with known Runge-Kutta integrators (e.g. Euler, midpoint, RK44) and provide theoretical guarantees for convergence rates.
Main theorem:

For a $p$-flat, $(s+2)$-differentiable convex function, if we discretize the ODE with order-$s$ Runge-Kutta integrator, we have

$$f(x(t)) - f(x^*) = O(t^{-\frac{ps}{s+1}})$$
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$p$-flat:

- $p = 2$: Gradient is Lipschitz continuous.
- $p = 4$: $\|x\|_4^4$
- $p = N$: $\log(e^{-x})$
Main theorem:

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\( p \)-flat:

\[
\begin{align*}
p = 2 : & \text{ Gradient is Lipschitz continuous.} \\
p = 4 : & \|x\|_4^4 \\
p = N : & \log(e^{-x})
\end{align*}
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Order-\( s \): Discretization error scales as \( O(h^{s+1}) \), \( h \) is the step size.
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\( p \)-flat:

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\begin{align*}
p = 2 & : \text{Gradient is Lipschitz continuous. } \\
p = 4 & : \|x\|_4^4 \\
p = N & : \log(e^{-x})
\end{align*}
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Order-\( s \): Discretization error scales as \( \mathcal{O}(h^{s+1}) \); \( h \) is the step size.
Our poster session:

Thu Dec 6th 05:00 -- 07:00 PM
Room 210 & 230 AB
Poster Number: 9