Differentially Private k-Means with Constant Multiplicative Error

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joint work with
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What is $k$-Means Clustering?

**Given:** Data points $S = (x_1, ..., x_n) \in (\mathbb{R}^d)^n$ and parameter $k$

Identify $k$ centers $C = (u_1, ..., u_k)$ minimizing $\text{cost}(C) = \sum_i \min_{\ell} \|x_i - u_\ell\|^2$
What is \( k \)-Means Clustering?

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Identify \( k \) centers \( C = (u_1, \ldots, u_k) \) minimizing \( \text{cost}(C) = \sum_i \min_{\ell} \|x_i - u_{\ell}\|^2 \)

✓ Probably the most well-studied clustering problem
✓ Tons of applications
✓ Super popular
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**What is Differentially Private \( k \)-Means?**

[Dwork, McSherry, Nissim, Smith 06] (informal)

- Every data point \( x_i \) represents the (private) information of one individual
- Goal: the output (the set of centers) does not reveal information that is specific to any single individual
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What is Differentially Private $k$-Means?

[Dwork, McSherry, Nissim, Smith 06] (informal)

✓ Every data point $x_i$ represents the (private) information of one individual

✓ Goal: the output (the set of centers) does not reveal information that is specific to any single individual

✓ Requirement: the output distribution is insensitive to any arbitrarily change of a single input point (an algorithm satisfying this requirement is differentially private)
What is $k$-Means Clustering?

Given:
- Data points $\mathcal{S} = \{x_1, \ldots, x_n\} \in \mathbb{R}^{d \times n}$
- Parameter $k$

Identify $k$ centers $\mathcal{C} = \{u_1, \ldots, u_k\}$ minimizing cost $\mathcal{C} = \text{minimize} \: \ell(x_i) - \ell(u)^2$.

What is Differentially Private $k$-Means?

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Why is that a good privacy definition?

Even if an observer knows all other data point but mine, and now she sees the outcome of the computation, then she still cannot learn “anything” on my data point.
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**Observe:** With privacy we must have additive error

- Assume $k = n = 3$
- OPT’s cost = 0
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- On at least one of these inputs our cost is $\approx \Lambda^2$
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$\Rightarrow$ We assume that input points come from the unit ball
## Previous and New Bounds

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<tr>
<th>Ref</th>
<th>Model</th>
<th>Runtime</th>
<th>Bounds</th>
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<tr>
<td>GLMRT’10</td>
<td>differential privacy</td>
<td>$n^d$</td>
<td>$O(1) \cdot \text{OPT} + \tilde{O}(k^2 \cdot d)$</td>
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<tr>
<td>NCBN’16</td>
<td>differential privacy</td>
<td>poly</td>
<td>$O(\log k) \cdot \text{OPT} + \tilde{O}(n)$</td>
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<td>FXZR’17</td>
<td>differential privacy</td>
<td>poly</td>
<td>$O(k \log n) \cdot \text{OPT} + \tilde{O}\left(k^{3/2} \cdot \sqrt{d}\right)$</td>
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<tr>
<td>BDLMZ’17</td>
<td>differential privacy</td>
<td>poly</td>
<td>$O(\log^3 n) \cdot \text{OPT} + \tilde{O}(k^2 + d)$</td>
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<tr>
<td>NS’18</td>
<td>differential privacy</td>
<td>poly</td>
<td>$O(k) \cdot \text{OPT} + \tilde{O}(k^{1.51} \cdot d^{0.51})$</td>
</tr>
<tr>
<td><strong>New</strong></td>
<td>differential privacy</td>
<td>poly</td>
<td>$O(1) \cdot \text{OPT} + \tilde{O}(k^{1.01} \cdot d^{0.51} + k^{3/2})$</td>
</tr>
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